University of Eswatini

Main Examination, 2020/2021

BASS, B.Ed, B.Sc

Title of Paper : Numerical Analysis II

Course Code : MAT411/M411

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. SECTION A(COMPULSORY): 40 MARKS
      Answer **ALL** QUESTIONS.
   b. SECTION B: 60 MARKS
      Answer ANY **THREE** questions.
      Submit solutions to ONLY THREE questions in Section B.
2. Each question in Section B is worth 20%.
3. Show all your working.
4. Special requirements: None

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A: Answer ALL Questions

A1. **a.** Show that $f(t, y) = ty^2$ satisfies a Lipschitz condition on the set $D = \{(t, y) : 1 \leq t \leq 2 \text{ and } 1 \leq y \leq 3\}$. [4]

b. Suppose the perturbation $\delta(t)$ is proportional to t, that is $\delta(t) = \delta t$ for some constant δ. Show directly that the following initial-value problem is well-posed; $y'(t) = y(t) + 1, \quad 0 \leq t \leq 1, \quad y(0) = 0.$ [6]

c. Derive the normal equations based on minimizing the least squares error $E(a_0, a_1, \ldots, a_n) = \int_a^b [f(x) - P_n(x)]^2 dx$ where $P_n(x) = \sum_{k=0}^{n} a_k x^k$. [6]

d. Derive the recurrence formula

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x),$$

where T_n are Chebyshev polynomials of order n defined by $T_n(x) = \cos(n \arccos(x))$, for each $n \geq 0$ with $x \in [-1,\ 1]$. [5]

e. Derive the **explicit** finite difference scheme for the heat equation, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$. [4]

f. Apply the Runge-Kutta method of fourth order in steps of 0.2, to solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$. [6]

g. For the initial value problem,

$$x' = -x + t + 1,\ 0 \leq t \leq 3,\ x(0) = 1,$$

approximate $x(0.1)$ by using one step of

i. Euler method, [3]

ii. Modified Euler method, [3]

iii. Taylor series method of order 2. [3]

Section B: Answer ANY 3 Questions

B2. (a) Use the Gram-Schmidt process to calculate L_1, L_2, where $\{L_0(x), L_1(x), L_2(x)\}$ is an orthogonal set of polynomials on $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ and $L_0(x) = 1$. The polynomials obtained from this procedure are called the Laguerre polynomials. [10]

(b) Use the Laguerre polynomials calculated above in (a) to compute the least squares polynomial of degree two on the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$ for $f(x) = x^2$. [10]

B3. (a) Solve by Taylor series method of third order the equation $y'(x) = (x^3 + xy^2)e^{-x}$, $y(0) = 1$ for y at $x = 0.1$, $x = 0.2$ and $x = 0.3$. [10]

(b) Use the Newton backward difference interpolating formula to derive 2-step Adam-Bashforth explicit formula. [10]

$$y_{i+1} = y_i + \frac{h}{2}[3f_i - f_{i-1}]$$

B4. Consider the following multi-step method for approximating the solution to an initial value problem,

$$y_{i+1} = 2y_i - y_{i-1} + \frac{h}{4}[f_{i-2} + 3f_{i-1}],$$
$$y_0 = a,\ y_1 = a_1,\ y_2 = a_2.$$

Discuss the stability, consistency and convergence of this method. [20]

B5. Consider the finite difference scheme;

$$U_j^{n+1} = U_j^n + \frac{k}{h^2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right) - kU_j^n,$$

for the numerical approximation of

$$u_t(x,t) = u_{xx}(x,t) - u(x,t), \quad 0 < x < 1,\ t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = f(x), \quad 0 \leq x \leq 1.$$

The constants k and h denote step sizes in the t and x variables, respectively.

(a) Find the local truncation error for this finite difference scheme. [10]

(b) Perform a Von-Neumann stability analysis for this scheme. [10]

B6. Consider the differential problem;

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0$$
$$u_x(0,t) = 1,\ u_x(1,t) = 0, \quad t > 0,$$
$$u(x,0) = x(1-x), \quad 0 \leq x \leq 1.$$

(a) Deduce the fully implicit numerical scheme resulting from using a **backward difference** approximation for the derivative u_t, and a **central difference** approximation for both u_x and u_{xx}. Show that the resulting finite difference equations may be written in matrix form as

$$\mathbf{A}\mathbf{U}^{(n)} + \mathbf{v} = \mathbf{U}^{(n-1)}$$

[12]

(b) Use this numerical scheme with $\Delta t = 0.1$ and $\Delta x = 0.5$ to derive the matrix equation to be solved to approximate $u(0.5, 0.1)$. Do not solve the equation. [8]

END OF EXAMINATION