University of Eswatini

Main Examination, 2020/2021

BASS IV, B.Ed (Sec.) IV, B.Sc. IV, B.Eng. III

Title of Paper : Partial Differential Equations

Course Number : MAT416/M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [40 Marks]: ANSWER ALL QUESTIONS

QUESTION A1 [40 Marks]

a) Consider the following equation with $-\infty < x < \infty$ and $t > 0$: [5]

$$u_{tt} - \pi^2 u_{xx} = 0, \quad u(x,0) = \arctan(\sin(x)), \quad u_t(x,0) = \frac{1}{\sqrt{9x^2 - 81}}.$$

Set up D'Alembert's equation that could be used to determine $u(x,t)$.

b) Consider the partial differential equation [7]

$$\begin{aligned} u_t &= u_{xx} \quad 0 < x < 100, \quad t > 0, \\ u(x,0) &= \tan(x), \quad 0 \le x \le 100, \\ u(0,t) &= 0, \quad u(100,t) = 0, \end{aligned}$$

Solve the resulting second order ordinary differential equation after implementing the method of separation of variables.

c) Use direct integration to solve for $u(\alpha, \beta)$ for [7]

$$u_\alpha = 2\alpha\beta e^{-2\beta}$$

d) Write down the expression for the Laplacian [5]

$$\mathcal{L}\left\{\frac{\partial^3 u}{\partial t^3}\right\}$$

where $u = u(x,t)$.

e) Find the half-range cosine expansion of $f(x) = x$, $0 \le x \le 2$. [7]

f) Write down a system that satisfies the temperature distribution $u(r,t)$ such that the initial temperature distribution of a thin unit circular disk is given by $2r^3$. The disk is allowed to cool down with its circular edge kept at temperature zero. [5]

g) Find the first derivative of [4]

$$y(t) = t^\nu J_\nu(t)$$

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

a) Consider the expression [10]

$$u - xy = f(x + y^2 - u^2)$$

where $u = u(x, y)$ and f is an arbitrary function. Find the partial differential equation for which the expression is a general solution.

b) Find the particular solution of [10]

$$yu_x + xu_y = u(x - y),$$

which contains the straight curve $u = 1$ on $y = x^2$.

QUESTION B3 [*20 Marks*]

a) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$: [4]

$$u_{tt} - 16u_{xx} = 0, \qquad u(x, 0) = 4, \qquad u_t(x, 0) = \sin(x).$$

Determine $u(x, t)$.

b) Consider the partial differential equation

$$8u_{yy} - 10u_{xy} + 2u_{xx} + u_x - u_y = 0.$$

(i) Determine whether the given partial differential equation is hyperbolic, parabolic or elliptic. [2]

(ii) Express the given partial differential equation in canonical form and hence find it's general solution. [14]

QUESTION B4 [*20 Marks*]

The initial temperature distribution of a thin circular disk is given by T_0. If the disk is allowed to cool down with its circular edge kept at temperature zero, the subsequent temperature distribution satisfies the system

$$\begin{aligned} k\left(u_{rr} + \frac{1}{r}u_r\right) &= u_t, \qquad 0 < r < 1, t > 0 \\ u(r, 0) &= T_0, \qquad 0 \leq r \leq 1, \\ u(1, t) &= 0, \qquad t \geq 0. \end{aligned}$$

Solve for $u(r, t)$. [20]

QUESTION B5 [*20 Marks*]

Find the solution of the steady-state problem [20]

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 < x < 1, \quad 0 < y < 1,, \\ u(0,y) &= u(1,y) = 0, \quad 0 \leq y \leq 1, \\ u(x,0) &= 0, \quad 0 \leq x \leq 1, \\ u(x,1) &= 5\sin(\pi x) - 8\sin(7\pi x), \quad 0 \leq x \leq 1. \end{aligned}$$

Determine the solution of the steady-state problem using the method of separation of variables.

QUESTION B6 [*20 Marks*]

a) Use Laplace transform to solve the system [12]

$$\begin{aligned} u_{xt} + \sin(t) &= 0, \quad x > 0, \quad t > 0, \\ u(x,0) &= x, \quad x \geq 0, \\ u(0,t) &= e^{-t}, \quad t \geq 0. \end{aligned}$$

b) Using the fact that the Laplace transform of $u(x,t)$ with respect to the variable t is given by

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt \equiv U(x,s),$$

show that

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0)$$

[8]

END OF EXAMINATION PAPER

$f(t)$	$\{f(t)\} = F(s)$	$f(t)$	$f(t) = F(s)$
1	$\frac{1}{s}$	$\frac{ae^{at} - be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
$e^{at}f(t)$	$F(s-a)$	te^{at}	$\frac{1}{(s-a)^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\delta(t)$	1	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$\delta(t-t_0)$	e^{-st_0}	$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$f'(t)$	$sF(s) - f(0)$	$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$f^n(t)$	$s^nF(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$	$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t^n \ (n = 0, 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$	$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$	$\frac{\sin at}{t}$	$\arctan\frac{a}{s}$
$\sin kt$	$\frac{k}{s^2+k^2}$	$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\cos kt$	$\frac{s}{s^2+k^2}$	$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
e^{at}	$\frac{1}{s-a}$	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$
$\sinh kt$	$\frac{k}{s^2-k^2}$		
$\cosh kt$	$\frac{s}{s^2-k^2}$		
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$		