University of Eswatini

Resit/Supplementary Examination, 2020/2021

BASS IV, B.Ed (Sec.) IV, B.Sc. IV, B.Eng. III

Title of Paper : Partial Differential Equations

Course Number : MAT416/M415

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

a) Given that the arbitrary function $F(x, y)$ is differentiable, show that [5]

$$u(x, y) = e^{-x/4}F(3x - 4y)$$

satisfies

$$4u_x + 3u_y + u = 0.$$

b) Determine the order of the partial differential equation satisfied by [3]

$$\rho(r, s) = \Omega(r - s) + \Psi(5r - s) + \Phi(5r + s),$$

where Ω, Ψ and Φ are arbitrary functions.

c) Suppose that the temperature distribution in a rod of length $8m$ is given by $T(x, t)$. Assuming that both ends are insulated such that there is no heat flow, write down a model that could be used to determine the temperature distribution $T(x, t)$, provided that the initial temperature distribution is given by e^{-2x}. [5]

d) Consider the wave equation

$$\begin{aligned}
\phi_{tt} &= c^2\phi_{xx}, \quad 0 < x < \pi, \quad t > 0, \\
\phi(x, 0) &= 3\sin(x), \quad 0 \leq x \leq \pi, \\
\phi_t(x, 0) &= 0, \\
\phi(0, t) &= \phi(\pi, t) = 0.
\end{aligned}$$

Write down the ordinary boundary value problems for $X(x)$ and $T(t)$ that must be solved in order to obtain the solution of the wave equation using the method of separation of variables. [6]

e) Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$u_{tt} - u_{xx} = 0, \quad u(x, 0) = 2x, \quad u_t(x, 0) = 3x^2.$$

Determine $u(0, 2)$. [7]

f) Solve the problem

$$u_x + xu_y = u, \quad u(x, 0) = u(0, y) = 5e^{8y},$$

using the method of characteristics [7]

g) Solve the problem

$$xu_x + u_t = x, \quad u(x, 0) = u(0, t) = 0,$$

using the method Laplace transforms [7]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

Consider the heat equation

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 4, \quad t > 0, \\ u(x,0) &= 8, \quad 0 \leq x \leq 4, \\ u_x(0,t) &= 0, \\ u(4,t) &= 0. \end{aligned}$$

Determine $u(x,t)$ using the method of separation of variables. [20]

QUESTION B3 [*20 Marks*]

(a) Consider Laplace's equation in a circle with Dirichlet boundary conditions.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} &= 0, \quad 0 < r < 2, \quad -\pi < \phi < \pi, \\ u(4,\phi) &= \phi, \quad -\pi < \phi < \pi. \end{aligned}$$

Is this a well defined problem? If yes, explain why and if no, explain why it is not a well defined problem. [3]

(b) Consider the Dirichlet problem of a sphere of radius a.

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial u}{\partial \phi}\right) &= 0, \quad 0 \leq r \leq a. \\ u(a,\phi) &= g(\phi), \quad 0 \leq \phi \leq \pi. \end{aligned}$$

Use the method of separation of variables to determine $u(r,\phi)$. [17]

QUESTION B4 [*20 Marks*]

Consider the Cauchy problem for the wave equation with $-\infty < x < \infty$ and $t > 0$:

$$\begin{aligned} p_{tt} &= v^2 p_{xx}, \\ p(x,0) &= \phi(x), \\ p_t(x,0) &= \psi(x), \end{aligned}$$

where v is a constant. Show that the solution of the wave equation is given by:

$$p(x,t) = \frac{1}{2}\left(\phi(x+vt) + \phi(x-vt) + \frac{1}{v}\int_{x-vt}^{x+vt} \psi(\gamma)d\gamma\right)$$

[20]

QUESTION B5 [*20 Marks*]

(a) Solve the first order partial differential equation [7]

$$\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = xu.$$

(b) Use Laplace transforms to find a solution [13]

$$\begin{aligned} u_{xx} - u_{tt} + \sin(\pi x) &= 0, \quad 0 < x < 1, \quad t > 0, \\ u(x,0) &= 0, \quad 0 \leq x \leq 1, \\ u_t(x,0) &= 0, \\ u(0,t) &= 0, \quad u(1,t) = 0. \end{aligned}$$

QUESTION B6 [*20 Marks*]

Consider the partial differential equation (PDE)

$$u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} - \cos(x)u_y + e^x = 0.$$

(a) Classify the partial differential equation by stating its order, linearity, homogeneity, and kind of coefficients. [4]

(b) Determine whether the given PDE is hyperbolic, parabolic or elliptic. [4]

(c) Express the partial differential equation

$$4u_{xx} + 5u_{xy} + u_{yy} + u_y + u_x = 0.$$

in canonical form. [12]

END OF EXAMINATION PAPER

$f(t)$	$f(t) = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
e^{at}	$\frac{1}{s-a}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$

$f(t)$	$f(t) = F(s)$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$\frac{\sin at}{t}$	$\arctan\frac{a}{s}$
$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$