UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2020/2021

**BASS, B.Ed (Sec.), B.Sc.**

**Title of Paper** : Optimisation Theory

**Course Number** : MAT418

**Time Allowed** : Three (3) Hours

**Instructions**

1. This paper consists of SEVEN (7) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, A2, B3, ..., B7) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Some formulas are given on the last page.

**Special Requirements: NONE**

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

## SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

### QUESTION A1 [20 Marks]

(a) Give precise definitions of the following.

(i) Convex set $S$ in $\mathbb{R}^n$. (2)

(ii) Convex function from a convex set $S \subseteq \mathbb{R}^n$ to $\mathbb{R}$. (2)

(iii) Concave function from a convex set $S \subseteq \mathbb{R}^n$ to $\mathbb{R}$. (2)

(b) Let $f$ be a function of $n$ variables and $g$ be a function of one variable. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, define the function $h$ of $n+1$ variables as follows:

$$h(x, y) = f(x) + g(y).$$

Show that if $f$ and $g$ are both convex functions, then $h$ is also a convex function. (5)

(c) Let $f(x_1, x_2) = 12x_1^{1/3}x_2^{2/3}$. Show that $f$ is concave for $x_1 > 0$ and $x_2 > 0$. (4)

(d) Let $f(x_1, x_2) = x_1^3 + 2x_1^2 + 2x_1x_2 + (1/2)x_2^2 - 8x_1 - 2x_2 - 8$. Find the range of values of $(x_1, x_2)$ for which $f$ is convex, if any. (5)

### QUESTION A2 [20 Marks]

(a) Consider the following linear programming problem.

$$\begin{array}{llrcrcl} \text{maximise} & z = & 3x_1 & + & 2x_2 & & \\ \text{subject to:} & & x_1 & & & \leq & 4 \\ & & x_1 & + & 3x_2 & \leq & 15 \\ & & 2x_1 & + & x_2 & \leq & 10 \\ & & & & x_1, x_2 & \geq & 0 \end{array}$$

(i) Use the graphical method to solve the problem. (6)

(ii) Write down the dual of the problem. (4)

(b) Use the Big-$M$ method to solve the following linear programming problem.

$$\begin{array}{llrcrcl} \text{maximise} & z = & x_1 & + & 2x_2 & & \\ \text{subject to:} & & 3x_1 & + & 2x_2 & \leq & 24 \\ & & x_1 & - & x_2 & \geq & 3 \\ & & & & x_1, x_2 & \geq & 0 \end{array}$$

(10)

## SECTION B: ANSWER ANY *THREE* QUESTIONS

**QUESTION B3 [20 Marks]**

Consider the following linear programming problem.

$$\begin{array}{lrcrcrcrcl} \text{maximise} & z & = & 10x_1 & + & 15x_2 & + & 4x_3 & & \\ \text{subject to:} & & & x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\ & & & 2x_1 & + & x_2 & + & x_3 & \leq & 4 \\ & & & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

(a) Find the dual of the problem. (5)

(b) Use the graphical method to solve the dual of the LP. (7)

(c) Use complementary slackness to solve the primal problem. (8)

**QUESTION B4 [20 Marks]**

(a) Find all local extrema and saddle points of the function

$$f(x_1, x_2) = x_1^4 + x_2^4 - 4x_1x_2 + 1.$$

(10)

(b) Use Golden Section Search to determine, within an interval of length 0.2, the optimal solution to

$$\begin{array}{rl} \text{maximise} & f(x) = -(x-2)^2 + 1 \\ \text{subject to:} & 1.75 \leq x \leq 2.25. \end{array}$$

(10)

**QUESTION B5 [20 Marks]**

Use the Kuhn-Tucker conditions to find the optimal solution to the following problem.

$$\begin{array}{lcl} \text{minimise} \quad z & = & (x_1-2)^2 + (x_2-4)^2 \\ \text{subject to:} & & x_1 + x_2 \leq 4 \\ & & x_1 + 3x_2 \leq 9 \end{array}$$

**QUESTION B6 [20 Marks]**

(a) Find the optimal solution to the following problem:

$$\begin{aligned} \text{minimise} \quad & f(x_1, x_2) = x_1^2 + x_2^2 + x_3^2 \\ \text{subject to:} \quad & x_1 + \ \ x_2 + \ \ x_3 = 9 \\ & x_1 + 2x_2 + 3x_3 = 20. \end{aligned}$$

(10)

(b) The Douglas-Cobb model says that when a company invests $L$ units of labour and $K$ units of capital, the production level $P$ is given by

$$P = bL^{\alpha}K^{1-\alpha}$$

where $b > 0$ and $0 < \alpha < 1$ are constants. Suppose that the cost per unit labour is $m$ emalangeni and the cost per unit capital is $n$ emalangeni and that the company has a budget of $B$ emalangeni to spend on total labour and capital. Show that maximum production occurs when

$$L = \frac{\alpha B}{m} \quad \text{and} \quad K = \frac{(1-\alpha)B}{n}.$$

(10)

**QUESTION B7 [20 Marks]**

(a) Use the method of steepest ascent to approximate the solution to

$$\begin{aligned} \text{maximise} \quad f(x_1, x_2) \quad &= \quad -5 + 4x_1 + 2x_2 - x_1^2 - x_2^2 \\ \text{subject to:} \quad & \quad\quad (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

Start at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$. (10)

(b) Perform *one* iteration of the feasible directions method on the following problem.

$$\begin{aligned} \text{maximise} \quad & f(x_1, x_2) = -(x_1 - 1)^2 - x_2^2 \\ \text{subject to:} \quad & 3x_1 + 4x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Begin at the point $(0, 1)$. (10)

END OF EXAMINATION PAPER