UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2020/2021

B.Sc IV, BASS IV

Title of Paper : Abstract Algebra II

Course Number : M423/MAT423

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1–A2, B3 – B7) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*20 Marks*]

A1 (a) Let $\mathbb{Z}$ denote set of integers. Define the operation addition "$\oplus$" and multiplication "$\odot$" on $\mathbb{Z}$ as follows. For any $a, b \in \mathbb{Z}$,

$$a \oplus b = a + b - 1 \qquad \text{and} \qquad a \odot b = a + b - ab.$$

Given that $(\mathbb{Z}, \oplus)$ is an abelian group and "$\odot$" is associative on $\mathbb{Z}$.
Prove that $(\mathbb{Z}, \oplus, \odot)$ is a commutative ring with identity. [12 marks]

(b) Compute the evaluation homomorphism $\varphi_3[(x^4 + 2x)(x^3 - 3x^2 + 3)]$ in $\mathbb{Z}_6$. [4 marks]

(c) By Fermate's Little Theorem, calculate the remainder of 2^{203} when divided by 13. [4 marks]

QUESTION A2 [*20 Marks*]

A2 (a) Let R be a ring. Define the following terms in R

i. an integral domain [3 marks]

ii. a division ring [3 marks]

iii. a field. [3 marks]

(b) Give example of

i. an integral domain that is not a field. [2 marks]

ii. a division ring that is not a field. [2 marks]

(c) Prove that every finite integral domain is a field. [7 marks]

SECTION B [*60 Marks*]: ANSWER ANY *THREE* QUESTIONS

QUESTION B3 [*20 Marks*]

B3 (a) Let $\mathbb{Z}$ be set of integers and $\mathbb{R}$ set of real numbers, if $\mathbb{Z}(\sqrt{2}) := \{a + \sqrt{2}b : a, b \in \mathbb{Z}\} \subset \mathbb{R}$. Prove that $\mathbb{Z}(\sqrt{2})$ is a subring of $\mathbb{R}$. [7 marks]

(b) Let R be an integral domain. Prove that a group of units $U(R)[x]$ in the polynomial ring $R[x]$ is just the set of units $U(R)$ in R. [7 marks]

(c) Show that $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a nilpotent element in $M_3(\mathbb{R})$. [6 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Define a unit in a ring R. And prove that a unit cannot be a zero divisor. [8 marks]

(b) Let $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : \quad a, b \in \mathbb{Z} \right\}$ be a subset of $M_2(\mathbb{Z})$. Prove that S is an additive subgroup of $M_2(\mathbb{Z})$ and also S is a right-ideal but not a left ideal. [12 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Find all the solutions of $x^2 + 2x + 4 = 0$ in $\mathbb{Z}_6$. [5 marks]

(b) Let $\mathbb{Z}_6$ be a ring and $I = \{0, 3\}$ be ideal of $\mathbb{Z}_6$. List the elements of $\mathbb{Z}/I$ and compute the addition and multiplication tables of $\mathbb{Z}_6/I$. [8 marks]

(c) By Fermate's Little Theorem, evaluate

$$2^{20} + 3^{30} + 4^{40} + 5^{50} + 6^{60} \mod 7.$$

[7 marks]

QUESTION B6 [*20 Marks*]

B6 (a) Let R be a ring and $U(R)$ a unit ring. Prove that

$$\sigma_u : R \to R, \quad \sigma_u(r) = uru^{-1}, \quad r \in R,$$

is an automorphism of R called an inner automorphism. [7 marks]

(b) Find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $f(x) = 3x^3$ is divided by $ix^2 + 5x + 2$ in $\mathbb{C}[x]$. [8 marks]

(c) Use Eisentein's criterion to show that $f(x) = 3x^4 - 10x^2 - 5x + 15$ is irreducible over $\mathbb{Q}$. [5 marks]

QUESTION B7 [*20 Marks*]

B7 (a) Prove that any prime element of an integral domain is irreducible. [6 marks]

(b) Define Unique Factorisation Domain (UFD). [5 marks]

(c) Find $d = \gcd(a, b)$ and x, y such that $d = ax + by$ if $a = 32 + 9i$ and $b = 4 + 11i$ in $\mathbb{Z}[i]$. [9 marks]

END OF EXAMINATION