UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2020/2021

BASS IV, B.Ed. (Sec.) IV; B.Sc IV

Title of Paper : Metric Space

Course Number : M431/MAT434

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Show all your working.
5. Start each new major question (A1, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
6. You can answer questions in any order.
7. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1 [*40 Marks*]

A1 (a) i. Define a metric space. [5 marks]

ii. Let $X = C[-1,1] := \{f : [-1,1] \to \mathbb{R} : f \text{ is continuous}\}$.
Define $d_2 : X \times X \to [0,\infty)$ for all $f, g \in X$ by

$$d_2(f,g) = \sqrt{\int_{-1}^{1} (f(x) - g(x))^2 dx},$$

for all $x \in [-1,1]$. Compute $d_2(f,g)$ given that

A. $f(x) = x^2$ and $g(x) = \frac{1}{2}x$; [3 marks]

B. $f(x) = \sin \pi x$ and $g(x) = 0$. [4 marks]

(b) Let (X,d) be a metric space and $A \subset X$. Define the following terms:

i. an open ball in X. [2 marks]

ii. A is an open set in X. [2 marks]

iii. limit point of A in X. [2 marks]

iv. Let $X = \mathbb{R}$ be the set of real numbers. Consider a usual metric $d : X \times X \to [0,\infty)$ defined by $d(x,y) = |x - y|$, for all $x, y \in X$. Compute $B_1(-2)$. [3 marks]

v. Let $X = \mathbb{R}$ be endowed with the usual metric. Let $A = [0,1)$. Show that A is not open in X. [3 marks]

(c) Let (X,d) be a metric space.

i. When is (X,d) a complete metric space? [2 marks]

ii. Define a contraction mapping on (X,d). [3 marks]

iii. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x,y,z) = \left(\frac{z}{3}, \frac{y}{2}, \frac{x}{2}\right)$. Show that f is a contraction on $\mathbb{R}^3$ with respect to the Euclidean metric space. [4 marks]

iv. State, without proof, the Banach Contraction Principle. [3 marks]

v. Fully explain, why the Banach Contraction Principle, fails to hold on the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{3}(2x + 6), \text{ for all } x \in [0,1].$$

[4 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2 [*20 Marks*]

B2 (a) Let $X = (0, \infty)$, prove that $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right| + |x^2 - y^2|$$

is a metric on X. [6 marks]

(b) Let $X = \mathbb{R}$ set of real numbers. Is $d^* : X \times X \to [0, \infty)$ defined by $d^*(x, y) = |x^2 - y^2|$ a metric on X? Justify your claim. [2 marks]

(c) Let (X, d) and (Y, ρ) be two metric spaces. Define an isometric map f from X into Y. [3 marks]

(d) Prove that the Inverse of a surjective isometry mapping is an isometry mapping. [5 marks]

(e) Consider the usual metric space $(\mathbb{R}, d)$ and the Euclidean space $(\mathbb{R}^3, d_3)$. Prove that the inclusion map $g : \mathbb{R} \to \mathbb{R}^3$ defined by

$$g(x) = (0, x, 0), \text{ for all } x \in \mathbb{R}$$

is an isometry. [4 marks]

QUESTION B3 [*20 Marks*]

B3 (a) Let (Y, d) be a complete metric space and $B \subseteq Y$. Assuming that B is complete, prove that B is closed. [6 marks]

(b) Let $X = \mathbb{R}$ with metric ρ_0 defined by

$$\rho_0(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

for arbitrary $x, y \in \mathbb{R}$. Find the following open balls:
(i) $B_{\frac{3}{2}}(4)$; (ii) $B_1(5)$; (iii) $S_2(3)$. [3,3,3 marks]

(c) Prove that an arbitrary union of open sets in X is open in X. [5 marks]

QUESTION B4 [*20 Marks*]

B4 (a) Let (X, ρ_X) and (Y, ρ_Y) be any two metric spaces. Define a continuous mapping $f : (X, \rho_X) \to (Y, \rho_Y)$. [5 marks]

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } x^2 + y^2 \neq 0 \\ 0, & \text{for } x = y = 0 \end{cases}$$

Prove that f is continuous at $(0, 0)$. [8 marks]

(c) Prove that image of a compact set under a continuous map is compact. [7 marks]

QUESTION B5 [*20 Marks*]

B5 (a) Let (X, d) be a metric space and for any $x, y, w, z \in X$.
Prove that

$$\left|d(x, y) - d(w, z)\right| \leq d(w, x) + d(z, y).$$

[5 marks]

(b) Let X be a nonempty set and suppose that (X, d) and (X, ρ) are metric spaces. Under what condition is d and ρ said to be an equivalent metric spaces? [3 marks]

(c) In $\mathbb{R}^n$, prove that the metrics $d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$ and $d_\infty(x, y) = \max\limits_{1 \leq i \leq n} |x_i - y_i|$, where $x := (x_1, x_2, \cdots, x_n)$, $y := (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ are equivalent. [8 marks]

(d) Let K be a subset of a metric space X. Under what condition is K compact?[4 marks]

QUESTION B6 [*20 Marks*]

B6 (a) Let $X = \mathbb{R}$ be endowed with discrete metric defined $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

for all $x, y \in X$. Let $A = [\frac{1}{2}, 2) \cup (3, 5]$. Compute

i. Limit of A,
ii. $\bar{A}$,
iii. $int(A)$,
iv. ∂A.

[3,3,3,3 marks]

(b) By using Euclidean metric, find the limit of the sequence

$$x_n := \left(\frac{1}{n^2}, \frac{n}{n+1}\right).$$

[4 marks]

(c) Define a homeomorphism. [4 marks]

END OF EXAMINATION