UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2020/2021

M.Sc. Mathematical Modelling & Mathematical Finance

Title of Paper : Advanced Numerical Analaysis

Course Number : MAT601

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions.
2. Answer ANY FIVE (5) questions.
3. Show all your working.
4. Start each new major question (1, 2 – 6) on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1 [*20 Marks*]

a) Find the condition number of $g(x) = \frac{16x}{2x+4}$ at x. Discuss if $g(x)$ is ill-conditioned or well-conditioned. [5]

b) Let A be a non-singular square matrix. Prove that if A is a scalar multiple of an orthogonal matrix, then $\kappa_2(A) = 1$. [5]

c) Let $A = \begin{bmatrix} -1 & 2a \\ a & -1 \end{bmatrix}$. Determine for what values of a is the matrix A is ill-conditioned. [5]

d) Prove that for any orthogonal matrix P,

$$\kappa(PA) = \kappa(AP) = \kappa(A)$$

for any $n \times n$ matrix A. [5]

QUESTION 2 [*20 Marks*]

a) Consider the linear system

$$\begin{bmatrix} 0.00001 & 3 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

i) Solve the system with partial pivoting using three decimal place accuracy. [10]

ii) Discuss the results obtained in the first part of this question and any possible solution to the problem encountered above. [5]

b) Determine whether the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 8 \end{bmatrix}$, is positive definite or not. [5]

QUESTION 3 [*20 Marks*]

a) Derive the classical Runge-Kutta method for [14]

$$\frac{dP}{dt} = \beta(t, P(t)).$$

b) Write down a scheme that could be used to solve the following system of differential equations

$$\frac{dR}{dt} = f_1(t, R, Q)$$
$$\frac{dQ}{dt} = f_2(t, R, Q).$$

using the classical RK-method. [6]

QUESTION 4 [*20 Marks*]

a) Consider the boundary value problem

$$a_1 u''(x) + a_2 u'(x) + a_3 u(x) = r(x), \qquad 0 < x < 1$$

subject to

$$u(a) = \alpha, \qquad u'(b) = \beta.$$

Using the centered difference approximation, determine the finite difference scheme that can be used to solve the boundary value problem. Express your scheme in matrix form indicating the implementation of the boundary conditions. [16]

b) Define consistency of a scheme. [4]

QUESTION 5 [*20 Marks*]

a) Given $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 1 & -5 \\ 3 & 1 & -2 \end{bmatrix}$,

i) find the LU decomposition of the tridiagonal matrix A. [6]

ii) and hence solve $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$. [4]

b) Determine the flop count of the LU decomposition. [10]

QUESTION 6 [*20 Marks*]

Consider the first order partial differential equation

$$u_t + 12u_x = 0, \quad (x,t) \in (0,1) \times (0,T), \quad u(t=0) = u_0(x)$$

with proper boundary conditions and initial condition. Assume that the domain $(0,1) \times (0,1)$ is partitioned into a uniform rectangular grid

a) Derive the Lax-Wendroff scheme for the differential equation. [6]

b) Use the von Neumann technique to determine the conditions under which the scheme is stable. [14]

QUESTION 7 [*20 Marks*]

Consider the 2-D parabolic differential equation

$$u_t = u_{xx} + u_{yy}, \quad (x, y, t) \in (0,1)^2 \times (0, T)$$

with proper boundary conditions and initial condition. Assume that the domain $(0,1) \times (0,1)$ is partitioned into a uniform rectangular grid

a) Determine Crank-Nicolson scheme for the 2-D parabolic differential equation. [6]

b) Use the von Neumann technique to determine the conditions under which the Crank-Nicolson scheme is stable. [14]

END OF EXAMINATION PAPER