UNIVERSITY OF ESWATINI

MAIN EXAMINATION, 2020/2021

M.Sc. in Mathematics

Title of Paper : Optimization

Course Number : MAT603

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions.
2. Answer ANY FIVE (5) questions.
3. Show all your working.
4. Start each new major question (Q1, Q2, ..., Q7) on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Question 1 [20 Marks]

(a) Give clear definitions of the following terms:

(i) A *convex set.* (2)

(ii) A *convex function.* (2)

(iii) A *concave function.* (2)

(b) Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function of many variables and let $k : \mathbb{R} \to \mathbb{R}$ be a function of one variable. Suppose h and k are both convex functions. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and let $y \in \mathbb{R}$. Define a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ as follows:

$$f(x, y) = h(x) + k(y).$$

Show that f is a convex function. (5)

(c) For each function below, determine whether it is concave, convex or neither on the given set.

(i) $f(x_1, x_2) = 2x_1^2 - 5x_1x_2 + 3x_2^2$ on $\mathbb{R}^2$. (3)

(ii) $f(x_1, x_2) = -x_1^2 - 2x_2^2$ on $\mathbb{R}^2$. (3)

(iii) $f(x_1, x_2) = x_1^2 - x_1x_2 + 2x_2^2$ on $\mathbb{R}^2$. (3)

Question 2 [20 Marks]

(a) Use the method of steepest descent to approximate the solution to the problem:

$$\text{minimise} \quad f(x_1, x_2) = (x_1 - 2)^2 + x_1 + x_2^2.$$

Begin at the point $\left(\frac{5}{2}, \frac{3}{2}\right)$. (8)

(b) Find all local maxima, local minima, and saddle points of

$$f(x_1, x_2) = 3x_1^2x_2 + x_2^3 - 3x_1^2 - 3x_2^2 + 2.$$

(12)

Question 3 [20 Marks]

(a) Use Lagrange multipliers to solve the following problem.

$$\begin{aligned} \text{minimise} \quad & f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 = 1. \end{aligned}$$

(10)

(b) Use K-T conditions to solve the following problem.

$$\begin{aligned} \text{minimise} \quad & f(x_1, x_2) = (x_1 - 4)^2 + (x_2 - 4)^2 \\ \text{subject to:} \quad & x_1 + \ x_2 \leq 4 \\ & x_1 + 3x_2 \leq 9. \end{aligned}$$

(10)

Question 4 [20 Marks]

(a) Write down the dual of the following linear programming problem without first converting it to a normal maximisation problem.

$$\begin{aligned} \text{maximise} \quad z = \; & x_1 - 2x_2 \\ \text{subject to:} \quad & x_1 + x_2 \leq 4, \\ & x_1 - x_2 = 4, \\ & x_1 \geq 1, \\ & x_1 \geq 0, x_2 \text{ urs.} \end{aligned}$$

(5)

(b) Consider the following linear programming problem:

$$\begin{aligned} \text{maximise} \quad & z = 3x_1 + 6x_2 + 2x_3 \\ \text{subject to:} \quad & 3x_1 + 4x_2 + 2x_3 \leq 200, \\ & x_1 + 3x_2 + 2x_3 \leq 100, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Find the dual and solve the dual graphically. Use complementary slackness to find the optimal solution to the primal. (15)

Question 5 [20 Marks]

(a) Consider the following linear programming problem:

$$\begin{array}{llrcrcrcl} \text{maximise} & z = & 2x_1 & - & x_2 & + & x_3 & & \\ \text{subject to:} & & 3x_1 & + & x_2 & + & x_3 & \leq & 50, \\ & & x_1 & - & x_2 & + & 2x_3 & \leq & 10, \\ & & x_1 & + & x_2 & - & x_3 & \leq & 20, \\ & & & & & & x_1, x_2, x_3 & \geq & 0. \end{array}$$

When the above problem is solved using the simplex method, it is found that in the optimal tableau, $x_{BV} = (x_2, x_1, s_3)^T$. Use the formulas to construct the optimal tableau of the LP. (8)

Hint: $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/4 & -3/4 & 0 \\ 1/4 & 1/4 & 0 \\ -1/2 & 1/2 & 1 \end{pmatrix}.$

(b) Consider the following linear programming problem with its optimal tableau.

$$\begin{array}{llrcrcl} \text{maximise} & z = & 3x_1 & + & 2x_2 & & \\ \text{subject to:} & & 2x_1 & + & x_2 & \leq & 100, \\ & & x_1 & + & x_2 & \leq & 80, \\ & & & & x_1, x_2 & \geq & 0. \end{array}$$

z	x_1	x_2	s_1	s_2	s_3	rhs
1	0	0	1	1	0	180
0	1	0	1	−1	0	20
0	0	1	−1	2	0	60

Answer the following questions.

(i) Find the range of values of c_1 (the objective function coefficient of x_1) for which the current BV remains optimal. (4)

(ii) Find the range of values of b_2 (the right-hand side of the second constraint) for which the current BV remains optimal. (4)

(iii) A third activity x_3 is being considered. If $c_3 = 3.5$ and $a_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, determine if it is worth introducing this activity. (4)

SOME FORMULAS: $\bar{c}_j = c_{BV}B^{-1}a_j - c_j$, $\bar{b} = B^{-1}b$, $\bar{a}_j = B^{-1}a_j$, $\bar{z} = c_{BV}B^{-1}b$,

Question 6 [20 Marks]

Consider the optimal control problem with state equation and cost function

$$\dot{x}_1 = x_1 + u_1, \quad J = \int_0^{t_1} \tfrac{1}{2} u_1^2 \, dt.$$

The initial and terminal states are $x_1(0) = X$ and $x_1(t_1) = 0$, respectively.

Use the Pontryagin Maximum Principle to find the optimal control, the optimal trajectory, and the optimal cost.

Question 7 [20 Marks]

Consider the controllable problem with state equations

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = 0, \quad |u_1| \leq 1.$$

(a) Find $\mathcal{C}(t_1, 0)$, the set of all points controllable to the origin in time t_1. (6)

(b) Find $\mathcal{C}(0)$, the set of all points controllable to the origin. (2)

(c) Find $\mathcal{R}(t_1, x^0)$, the set of reachable points from $x^0 = (p, q)^T$ in time t_1. (6)

(d) Write down the state equations for the time-reversed problem and verify that $\mathcal{R}(t_1, 0)$ for the time-reversed problem is the same as $\mathcal{C}(t_1, 0)$ for the original problem. (6)

END OF EXAMINATION