UNIVERSITY OF ESWATINI

2021 MAIN EXAMINATION

MSc

Title of Paper : POPULATION DYNAMICS & EPIDEMIOLOGY

Course Code : MAT606

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions.
2. Answer any FIVE (5) questions
3. Show all your working.
4. Start each new major question on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1 [20 MARKS]

(a) Explain what you understand by the term *bifurcation* as applied to a dynamical system. [2]

(b) Consider the first order system

$$\dot{x} = rx - \frac{x}{1+x^2},$$

where r is a bifurcation parameter.

(i) Find algebraic expressions for all the fixed points as r varies and state (in terms of r) the condition for the fixed point at the origin to be stable. [4]

(ii) Sketch a bifurcation diagram of the system. [4]

(b) Given the Rössler system;

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + ay,\\ \dot{z} &= b + z(x - c),\end{aligned}$$

where a, b, c are real and $a \neq 0$.

(i) which terms (if any) are nonlinear? [2]

(ii) Find all the fixed points in terms of a, b and c and state a condition for the existence of at least two fixed points. [6]

(iii) Write the matrix whose eigenvalues determine the stability of the fixed points. [2]

QUESTION 2 [20 MARKS]

A two-species predator-prey system with populations x and y is modelled by the equations

$$\begin{aligned}\frac{dx}{dt} &= Ax\left(1 - \frac{x}{k}\right) - Bxy\left(1 - e^{-cx}\right),\\ \frac{dy}{dt} &= -Dy + Ey\left(1 - e^{-cx}\right),\end{aligned}$$

where A, B, C, D and k are positive.

(a) Describe the ecological meaning of all the terms of the model. [4]

(b) By setting

$$X = \frac{x}{k}, \quad Y = \frac{By}{A}, \quad T = At, \quad \alpha = \frac{E}{A}, \quad \delta = \frac{D}{A}, \quad \beta = ck,$$

show that the system is transformed to

$$\begin{aligned}\frac{dX}{dT} &= X(1 - X) - XY\left(1 - e^{-\beta X}\right),\\ \frac{dY}{dT} &= -\delta Y + \alpha Y\left(1 - e^{-\beta X}\right).\end{aligned}$$

[6]

(c) Determine the co-ordinates of the three fixed points; noting any parameter restrictions. [3]

(c) Establish the conditions for stability of the fixed points. [7]

QUESTION 3 [20 MARKS]

The following system of differential equations model an interaction between two populations:

$$\begin{aligned}\dot{x} &= x(1-x) - \alpha xy,\\ \dot{y} &= \beta xy - \gamma y,\end{aligned}$$

where α, β and γ are strictly positive constants.

(a) Briefly describe the type of interaction that these equations might be model. [2]

(b) Find the equilibrium points of these equations and determine the values of the parameters for which they satisfy $x \geq 0$, $y \geq 0$ and the values for which they are asymptotically stable. [12]

(c) Sketch phase portraits of the system for the cases (a) $\gamma > \beta$ and (b) $\gamma < \beta$. [6]

QUESTION 4 [20 MARKS]

(a) State the Poincaré-Bendixson theorem. [6]

(b) Given the system

$$\begin{aligned}\dot{x} &= -x^3 - y,\\ \dot{y} &= -y^3 + x.\end{aligned}$$

(i) Show, using the Bendixson's Negative Criterion, that the system does not exhibit periodic solutions. [6]

(ii) Use a suitable Lyapunov function to show that the zero steady state is stable. [8]

QUESTION 5 [20 MARKS]

(a) (i) State the Dulac's criteria for the non-existence of periodic orbits. [4]

(ii) Use the function $\rho(x,y) = be^{-2\beta x}$ to show that the system of differential equations

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \alpha x^2 + \beta y^2 - ax - by,\end{aligned}$$

has no limit cycles in $\mathbb{R}^2$. [6]

(b) Consider the differential equations

$$\begin{aligned}\dot{x} &= x - y,\\ \dot{y} &= 1 - x^2.\end{aligned}$$

Sketch a phase portrait of the system. [10]

QUESTION 6 [20 MARKS]

Consider the SIRS model

$$\frac{dS}{dt} = -\beta IS + \gamma R,$$
$$\frac{dI}{dt} = \beta SI - \nu I,$$
$$\frac{dR}{dt} = \nu I - \gamma R.$$

(a) Describe all the terms in the model. [6]

(b) Determine R_0. [2]

(c) By letting $N = S+I+R$, reduce the system to three equations to a two equations model, determine all the equilibria and investigate the stability of the disease free equilibrium point. [10]

(d) Give an example of a disease which can have the same dynamics. [2]

QUESTION 7 [20 MARKS]

The following diagram represents the dynmics of an SIR epidemic model without births or deaths.

$$S \xrightarrow{\ \beta\ } I \xrightarrow{\ \gamma\ } R$$

where β and γ are positive constants.

(a) Write down the set of equations to describe these dynamics and explain why the system can be fully described using two of the equations. [6]

(b) (i) Define basic reproductive ratio, R_0, in words. [2]

(ii) Derive the expression for R_0 in terms of your model parameters. [4]

(c) Deermine the maximum number of infectives ($I_{\max}$). [8]

——— The end ———