UNIVERSITY OF ESWATINI

APRIL 2021 MAIN EXAMINATION

MSc in Mathematics

Title of Paper : Spectral Methods for Differential Equations

Course Number : MAT607

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions.
2. Answer ANY FOUR (4) questions.
3. Show all your working.
4. Start each new major question on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

QUESTION 1 [*25 Marks*]

The solution of the differential equation

$$xy''(x) - 3y'(x) = 0$$

with boundary conditions

$$y(0) = 1, \quad \text{and} \quad y(1) = 0$$

can be approximated by the polynomial

$$Y(x) = c_0 + c_0 x + c_2 x^2$$

Use spectral collocation points with equally spaced collocation points to show that matrix equation that results from the collocation process is $y(x) = 3x^2 - 2x$ [15 Marks]

QUESTION 2 [*25 Marks*]

Consider the Falkner-Skan boundary layer flow equation

$$f'''(x) + f(x)f''(x) + 1 - f'(x)^2 = 0$$

whose boundary conditions are

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1$$

(a) Derive the quasi-linearisation method scheme that can be used to iteratively solve the Falkner-Skan equation [10 Marks]

(b) Illustrate how the matrix approach of the spectral collocation can be applied on the quasi-linearisation method and boundary conditions with the transformations

$$f^{(n)}(x_i) = \sum_{k=0}^{N} \mathbf{D}_{i,k}^{(n)} f(z_k) = \mathbf{DF}, \quad i = 0, 1, 2 \ldots, N$$

where $\mathbf{D}$ is the differentiation matrix and $\mathbf{F}$ is the vector of unknowns at the so-called collocation points $z_i = \cos\left(\frac{\pi i}{N}\right)$. [15 Marks]

QUESTION 3 [*25 Marks*]

Consider the following linear partial differential equation with boundary

$$u_t = u_{xx}$$

with boundary conditions

$$u(-1,t) = f(t), \quad u(1,t) = g(t)$$

and initial condition

$$u(x,0) = h(x).$$

In approximating the solution of the differential equation, consider three equally spaced nodes x_0, x_1, x_2 in the space variable x and two nodes t_0, t_1 in the time variable t. The approximating function corresponding to the selected nodes is

$$U(x,t) = c_{0,0} + c_{1,0}x + c_{2,0}x^2 + c_{0,1}t + c_{1,1}tx + c_{2,1}tx^2.$$

Give a detailed description of the collocation process that can be used to solve the PDE using the approximating polynomial given above.

QUESTION 4 [*25 Marks*]

Consider the linear system of ordinary differential equations

$$u'' - 2w' + u = 0$$
$$w'' - 2u' + w = 0$$

subject to the following boundary conditions

$$u(-1) = \sinh(1), \quad u'(1) - u(1) = \cosh(1)$$
$$w'(-1) + w(-1) = \sinh(1), \quad w'(1) - w(1) = \sinh(1)$$

Use the matrix based spectral collocation method with to illustrate how the linear system can be reduced to a matrix system of the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

In your illustration, give definitions of the matrices and vectors and demonstrate how the boundary conditions can be imposed on the matrices and vectors. [25 Marks]

QUESTION 5 [*25 Marks*]

Consider the regular eigenvalue problem

$$x^2 y'' + 3xy' + \lambda y = 0$$

subject to the boundary conditions

$$y\,(1) = 0 \quad y\,(2) = 0$$

Describe how the spectral quasi-linearisation method can be used to solve the eigenvalue problem [25 Marks]

QUESTION 6 [*25 Marks*]

Consider the linear ordinary differential equation

$$V^{iv} - V'' + VV''' - V'V'' = 0$$

with boundary conditions

$$V(0) = 0, \;\; V'(0) = 0, \;\; V(1) = 1, \;\; V'(1) = 0$$

Give a sketch of the Matlab code that can be used to solve the differential equation using a function, say *cheb.m* for invoking the collocation points x and differentiation matrix D. Your code sketch must include a line for plotting the residual error profile. [25 Marks]

END OF EXAMINATION PAPER