University of Eswatini

Final Examination, 2021

MSc Applied Mathematics

Title of Paper : Special Topics in Mathematical Modelling

Course Code : MAT612

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of TWO sections.
   a. **SECTION A: (80 MARKS)** Answer any four (4) questions from this section.
   b. **SECTION B: (20 MARKS)** Answer one question from this section.

2. Each question is worth 20%.

3. Show all your working.

4. Special requirements: None

THIS PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

Section A: Answer four questions from this section.

QUESTION 1

a. Find **second order** perturbation approximate solutions for both roots of the following quadratic equation.
$x^2 - (3 + 2\epsilon)x + 2 + \epsilon = 0$ [12]

b. For $\epsilon x^3 + x + 1 = 0$, find the first order approximation for the first root, then for the two roots, derive the regular perturbation equation to be solved to find them, do not find the roots. [8]

QUESTION 2

a. Use regular perturbation to find the **second order** approximate solution for the initial value problem;

$$y'' = -\epsilon y' - 1, \quad y(0) = 0, \quad y'(0) = 1.$$

[8]

b. Consider the following second order boundary value equation;

$$\varepsilon y'' + 2y' + y^3 = 0, \quad y(0) = 0, \quad y(1) = \frac{1}{2}.$$

Assume that the boundary layer occurs at $x = 0$. Use the method of Dominant Balance to find a zeroth order approximate solution $y(x, \varepsilon)$ for the equation. [12]

QUESTION 3

a. Consider the following second order boundary value equation;

$$-\varepsilon y'' + y' + y = 1, \quad y(0) = 0, \quad y(1) = 0.$$

Assume that the boundary layer occurs at $x = 1$. Use the method of Dominant Balance to find a zeroth order approximate solution $y(x, \varepsilon)$ for the equation. [14]

b. For $\epsilon x^2 - 2x - 1 = 0$, use regular perturbation to find the first order approximation for the first root, then derive the regular perturbation equation to be solved to find the second root, do not solve. [6]

QUESTION 4

Given that the following boundary layer problem has boundary layer behavior at both ends of the domain, find a composite expansion of the solution on [0, 1].

$\epsilon y'' + \epsilon(x+1)^2 y' - y = x - 1, \quad y(0) = 0, \quad y(1) = -1.$ [20]

QUESTION 5

Apply the **Homotopy Perturbation Method** to find 2^{nd} order approximate solution for the following equations;

a. $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}$ with initial condition $u(x,0) = 2x$, and boundary conditions

$$u(0,t) = 0, \quad u_x(0,t) = \frac{2}{1+2t}.$$

[8]

b. $u_{tt} = \frac{1}{2}y^2 u_{xx} + \frac{1}{2}x^2 u_{yy}, \quad 0 < x,\ y < 1,\ t > 0$ with the boundary conditions

$$u(0,y,t) = y^2 e^{-t},\ u(1,y,t) = (1+y^2)e^{-t},\ u(x,0,t) = y^2 e^{-t},\ u(x,1,t) = (1+x^2)e^{-t},$$

and the initial conditions

$$u(x,y,0) = x^2 + y^2,\ u_t(x,y,0) = -(x^2 + y^2).$$

[12]

Section B: Answer ONE Question from this section

QUESTION 7

A public health organization engaged a mathematical scientist and they were interested in a mathematical model that will enhance their understanding of the dynamics of COVID-19 patients with the interactions between those who are at low risk and those who are high risk. The mathematical scientist brought out the following as a summary to what must be done.

We divide our model into eight compartments namely S_h,S_l, E_h,E_l,I_h,I_l,H,R, for high risk Susceptible population, low risk Susceptible population, high risk Exposed population, low risk Exposed population, high risk Infected population, low risk Infected population, Hospital population, and Recovered population, respectively. The main assumption is that all constants are positive. We assume that the high risk susceptible population become exposed after interacting with the high risk infected population, low risk infected population, and with those who are hospitalized. Similarly, those who are low risk susceptible population become exposed after interacting with the high risk infected population, low risk infected population, and with those who are hospitalized. In both cases, the infection rate β_h and β_h are influenced by other non-pharmaceutical interventions like wearing masks, maintaining social distance and washing hands regularly at the rate ϵ_h and ϵ_l.

The exposed population (both high risk and low risk) progress to the infectious classes at the rate k_h and k_l respectively. Infected individuals (both high risk and low risk) can either be hospitalized or recover at the rate σ_h and σ_l respectively. The fraction of those that get hospitalized are denoted by ρ_h and ρ_l from the high risk infectious class and low risk infectious class respectively. The hospitalized recover at the rate α. The infectious individuals die due to the disease at the rate δ_h and δ_l respectively for high risk and low risk. The deceased from the disease and have been hospitalized are denoted by δ. The high risk susceptible population progresses to the low risk susceptible population at the rate m_h. High risk susceptible individuals may become low risk if they are follow all the protocols to prevent the spread of Covid-19 to the core. Low risk susceptible individuals progresses to the high risk susceptible population at the rate m_l. A low risk susceptible individual may become high risk if this Covid-19 era, they are diagnosed with diabetes, chronic respiratory and cardiovascular diseases.

1. Write down the flow model. [10]
2. Write down the nonlinear first order ordinary differential equation model. [10]

QUESTION 8

Consider the single-species model with Allee effect and Haverst.

1. Write down the single-species model with Allee effect and constant harvest. [5]
2. Derive the three equilibria of the single-species model with Allee effect and constant harvest. [15]

END OF EXAMINATION