UNIVERSITY OF ESWATINI

MAIN EXAMINATION 1, 2020/2021

M.Sc. Mathematics 1

Title of Paper : STOCHASTIC DIFFERENTIAL EQUATIONS

Course Number : MAT 632

Time Allowed : Three (3) Hours

Instructions:

1. This paper consists of SIX (6) questions in TWO sections.
2. Section A is **COMPULSORY** and is worth 40%. Answer ALL questions in this section.
3. Section B consists of FIVE questions, each worth 20%. Answer ANY THREE (3) questions in this section.
4. Start each new major question (A1-A4, B2 – B6) on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

SECTION A [*40 Marks*]: ANSWER ALL QUESTIONS

QUESTION A1.

(a.) State and prove Borel-Cantelli Lemma. [10 marks]

QUESTION A2.

(a.) Define a Borel σ-algebra generated by $\mathfrak{U} \subseteq \Omega$. [3 marks]

(b.) Suppose $G_1, G_2, G_3, ..., G_n$ are disjoint subsets of Ω such that $\Omega = \bigcup_{i=1}^{n} G_i$. Prove that a family $\mathbf{G}$ consisting of $\emptyset$ and all unions of $G_1, G_2, G_3, ..., G_n$ constitute a $\sigma-$ algebra on Ω. [7 marks]

QUESTION A3.

(a.) Evaluate $I = \int_0^T B(t)dB(t)$ [4 marks]

(b.) Evaluate:

i. $E[B_t^8]$ [3 marks]

ii. $E[B_t^{26}]$ [3 marks].

QUESTION A4.

Write $Z(t)$ in the form

$$dZ(t) = \mu(t,\omega)dt + \sigma(t,\omega)dB(t)$$

(i.) $Z(t) = t^5 B^3(t)$. [3 marks]

(ii.) $Z(t) = t^7 + 3e^{2B(t)}$. [3 marks]

(iii.) $Z(t) = \ln(t)e^{2B(t)}$. [4 marks]

SECTION B: ANSWER ANY *THREE* QUESTIONS

QUESTION B2

a.) State the Ito representation theorem. [4 marks]

(b.) Prove that if

$$dZ(t) = Z(t)\theta(t,\omega)dB(t)$$

then $Z(t)$ is a martingale for all $t \leq T$ provided that
$Z(t)\theta_k(t,\omega) \in \nu(0,T) \quad 1 \leq k \leq n.$ [16 marks]

QUESTION B3

a.) Define an adapted process. [2 marks]

b.) Define the Ito integral. [5 marks]

c.) State and prove the Ito isometry for elementary and bounded function $\phi(t,\omega)$. [13 marks]

QUESTION B4.

(a.) State the Markov property for Ito diffusions. [4 marks]

(b.) Evaluate: $\int_0^T t^2 B^3(t) dB(t)$. [16 marks]

QUESTION B5.

(a.) Solve:

$$dX(t) = \alpha X(t)dt + \rho X(t)dB(t) - dH(t,\omega,X(t))$$

Given that $dH(t,\omega,X(t)) \sim kX(t)dt$. [15 marks]

(b.) Evaluate: $E[X(t)]$. [5 marks]

QUESTION B6.

(a.) Solve:

$$dX(t) = \alpha X(t)dt + \rho dB(t) \bullet H(t, \omega, X(t))$$

Given that $H(t, \omega, X(t)) \sim kX(t)$. [15 marks]

(b.) Evaluate: $\sigma^2(X(t))$. [5 marks]

END OF EXAMINATION