University of Eswatini

Final Semester I Examination, 2020/2021

M.Sc. Mathematics

Title of Paper : Advanced Applied Analysis

Course Number : MAT633

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SEVEN (7) questions. Answer ANY FIVE (5) questions.
2. You can answer questions in any order.
3. Indicate your program next to your student ID.

Special Requirements: NONE

This examination paper should not be opened until permission has been given by the invigilator.

QUESTION 1 [*20 Marks*]

1 (a) Suppose X is a vector space over a field $\mathbb{F}$.

i. Write down the conditions for X to be a normed vector space together with the real valued function $||.|| : X \to [0, \infty)$. [4 marks]

ii. Which properties determine X as a Banach space? [3 marks]

(b) Let $X = C[0,2]$, for all $f \in C[0,2]$ defined a norm on X by

$$||f||_1 = \int_0^2 e^x |f(x)| dx.$$

Evaluate $||f||_1$, if $f(x) = \frac{1}{2}x$. [4 marks]

(c) Prove that the real sequence space $\ell_p (1 \leq p < \infty)$ is complete. [9 marks]

QUESTION 2 [*20 Marks*]

2 (a) Determine the matrix norm subordinate to

i. the one norm and [3 marks]

ii. the infinity norm [3 marks]

for the following matrix:

$$B = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -3 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

(b) Show that the functional defined on continuous space $C[a,b]$ by

$$f(x) = \int_a^b x(t)k(t)dt$$

where ($k \in C[a,b]$ is fixed) is linear and bounded. [6 marks]

(c) Let X and Y be normed linear spaces over a field $\mathbb{F}$ and $T : X \to Y$ a linear map. Prove that T is continuous at some point in X if and only if T is bounded on X. [8 marks]

QUESTION 3 [*20 Marks*]

3 (a) Give a detailed explanation (Definition) what is meant by that the functions f_n *converge uniformly and pointwisely* to f on the interval $[a,b]$ as $n \to +\infty$. [6 marks]

(b) Let $f_n : [0, \frac{2}{3}] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $x \in [0, \frac{2}{3}]$. Prove that f_n converges uniformly to $f = 0$ in $[0, \frac{2}{3}]$. [5 marks]

(c) Let K be a compact metric space. Prove that $C(K)$ is complete. [9 marks]

QUESTION 4 [*20 Marks*]

4 (a) Let $X = C[0,1]$. Define $\langle .,. \rangle : X \times X \to \mathbb{C}$ for each $f, g \in X$ the inner product on X by $\langle f, g \rangle = \int_0^1 \overline{f(t)} g(t) dt$, where $\overline{f(t)}$ is the conjugate of $f(t)$. Compute $\langle .,. \rangle$, when $f(t) = g(t) = 1 + it$. [5 marks]

(b) Let $x, y \in X$ where X is an inner product space. Find $||y||$ if

$$||x|| = \sqrt{17}, \ ||x+y|| = 4 \ \text{and} \ ||x-y|| = 6.$$

[5 marks]

(c) Let $X = \mathbb{R}^3$. For any $x, y \in \mathbb{R}^3$, define, the inner product and norm on $\mathbb{R}^3$ by

$$\langle x, y \rangle = x^T y \ \text{and} \ ||x||_2 = \sqrt{\sum_{i=1}^{3} x_i^2}$$

respectively. Given a set of three linearly independent vectors in $\mathbb{R}^3$

$$x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \ x^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \ x^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Using the Gram Schmidt procedure, generate an orthonormal set. [10 marks]

QUESTION 5 [*20 Marks*]

5 (a) Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous function. Suppose that $\{f_n\}$ converges uniformly to some $f : [a,b] \to \mathbb{R}$ on $[a,b]$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

[12 marks]

(b) Let $f_n(x) = \frac{2n+\sin nx}{3n+\sin^2 nx}$, for $x \in \mathbb{R}$.
If $\{f_n\}$ converges uniformly on $\mathbb{R}$, then compute

$$\lim_{n\to\infty} \int_0^{3\pi} f_n(x)dx.$$

[8 marks]

QUESTION 6 [*20 Marks*]

6 (a) Let (X, d) be a metric space and $f : X \to X$ a mapping on X.

i. Give a detailed explanation (Definition) for f to be a contraction mapping on X. [4 marks]

ii. Prove that every continuous mapping is a contraction mapping. [5 marks]

(b) Let $T : C[0,1] \to C[0,1]$ be defined by

$$T(u)(x) = \frac{1}{2}\int_0^x u(s)ds$$

for all $u \in C[0,1]$ and for all $x \in [0,1]$. Prove that T is a contraction mapping on $C[0,1]$ with sup-metric. [6 marks]

(c) State the Contraction Mapping Principle. [5 marks]

QUESTION 7 [*20 Marks*]

7 (a) Find the least squares solution of the points $\{(1,2),(3,5),(4,5),(6,8),(6,9),(7,10)\}$ in $xy-$plane. [6 marks]

(b) Using Weierstrass M-Test, prove that the series

$$\sum_{n=1}^{\infty} \frac{x^3}{3+n^2x^2}, \ x \in \mathbb{R}$$

converges uniformly on $\mathbb{R}$. [5 marks]

(c) Let $1 < p, q < \infty$ be conjugate exponents, (X, Σ, μ) a measure, if $f \in L^p(X,\mu)$, $g \in L^q(X,\mu)$ with $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$fg \in L^1(X,\mu) \ \text{ and } \ \int_X |fg| d\mu \leq ||f||_p ||g||_q.$$

[9 marks]

END OF EXAMINATION