UNIVERSITY OF SWAZILAND

MAINL EXAMINATION, 2020/2021

MSc.I

Title of Paper : Advanced Statistics

Course Number : MAT 636

Time Allowed : Three (3) Hours

Instructions

1. This paper consists of SIX (6) questions.
2. Answer ANY FOUR (4) questions in this section.
3. Show all your working.
4. Start each new major question on a new page and clearly indicate the question number at the top of the page.
5. You can answer questions in any order.
6. Indicate your program next to your student ID.

Special Requirements: NONE

THIS EXAMINATION PAPER SHOULD NOT BE OPENED UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

ANSWER ANY *FOUR* QUESTIONS

QUESTION A1 [*25 Marks*]

A1 (a) Let Y be a binomial random variable based on m trials and success probability θ. Let $E(Y) = m\theta$, show that

$$\sigma^2 = m\theta(1-\theta)$$

[8 Marks]

(b) The random variable X has the Normal distribution with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \quad -\infty < x < \infty$$

(i) Show that X has moment generating function $M_X(t) = \exp(\mu t + \frac{\sigma}{2}t^2)$.

[10 Marks]

(ii) For constants a and b, show that the moment generating function of $Y = aX + b$ is

$$e^{bt}M_X(at).$$

Use this result to obtain the moment generating function of

$$Z = \frac{X-\mu}{\sigma}.$$

Deduce the distribution of Z.

[7 Marks]

QUESTION A2 [*25 Marks*]

A2 (a) Let $X_1, \ldots, X_n$ be a random sample of size n from the exponential distribution whose pdf is

$$f(x,\theta) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) Use the method of moments to find a point estimator for θ.

[8 Marks]

(ii) The following data represent the time intervals between the emissions of beta particles: Assuming

0.9	0.1	0.1	0.8	0.9	0.1	0.1	0.7	1.0	0.2
0.1	0.1	0.1	2.3	0.8	0.3	0.2	0.1	1.0	0.9
0.1	0.5	0.4	0.6	0.2	0.4	0.2	0.1	0.8	0.2
0.5	3.0	1.0	0.5	0.2	2.0	1.7	0.1	0.3	0.1
0.4	0.5	0.8	0.1	0.1	1.7	0.1	0.2	0.3	0.1

the data follow an exponential distribution, use a(i) to compute a moment estimate for the parameter θ.

[5 Marks]

(b) Consider the experiment of tossing a fair coin 3 times. Let X be the number of heads on the first toss and F the number of heads on the first two tosses. Fill the joint probability table for X and F. Compute Cov(X, F).

[12 Marks]

QUESTION A3 [*25 Marks*]

A3 (a) Suppose the joint probability density function (X_1, X_2) is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} 1, & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $Y_1 = X_1 + X_2$ and $Y_1 = X_1 - X_2$.

i. Find the joint probability density function of (Y_1, Y_2)

[8 Marks]

ii. Compute $E(Y_2)$.

[5 Marks]

(b) Let $Y_1, \ldots, Y_n$ be a random sample from a population with pdf

$$fy) = \begin{cases} \frac{1}{\alpha} y^{(1-\alpha)/\alpha}, & \text{for } 0 < y < 1; \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that the maximum likelihood estimator of α is $\hat{\alpha} = -(1/n)\sum_{i=1}^{n} \ln(Y_i)$.

[6 Marks]

(ii) Is $\hat{\alpha}$ a consistent estimator of α?

[5 Marks]

QUESTION A4 [*25 Marks*]

A4 (a) Let X be any continuous random variable, and let $F(x)$ denote its cumulative distribution function. Suppose that U is a continuous random variable with the uniform distribution on the interval 0 to 1, and define the new random variable Y by

$$Y = F^{-1}(U),$$

where $F^{-1}(.)$ is the inverse function of $F(.)$.

(i) By considering the cumulative distribution function of Y, show that Y has the same distribution as X.

[8 Marks]

(ii) Briefly describe a method of simulating pseudo-random variates from a continuous probability distribution, based on this result.

[6 Marks]

(b) Now suppose $f(y, \theta) = \frac{1}{\theta} e^{-y/\theta}, x > 0$.

(i) State the factorisation criterion for sufficient statistics and use it to find a sufficient for θ.

[5 Marks]

(ii) Show that the sufficient estimator found in A4b(i) is an unbiased estimator of θ.

[6 Marks]

QUESTION A5 [*25 Marks*]

A5 (a) In Bayesian inference define what is meant by a conjugate prior distribution.

[4 Marks]

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Bernoulli distribution where

$$P(Y_i = 1) = p \text{ and } P(Y_i = 0) = 1 - p,$$

and assume that the prior distribution for p is $beta(\alpha, \beta)$, i.e.

$$f(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the posterior distribution for p.

[15 Marks]

(ii) Find the Bayes estimators for p for $\alpha = 10, \beta = 30, n = 25$, and $\sum y_i = 10$.

[6 Marks]

QUESTION A6 [*25 Marks*]

A6 (a) A student examined the effect of varying the water/cement ratio on the strength of concrete that had been aged 28 days. For concrete with a cement content of 200 pounds per cubic yard, the student obtained the data presented in the Table below.

Water/Cement ratio	Strength (100 ft/lb)
1.21	1.302
1.29	1.231
1.37	1.061
1.46	1.040
1.62	0.803
1.79	0.711

Let Y denote the strength and x denote the water/cement ratio.

(i) Fit the model $E(Y) = \beta_0 + \beta_1 x$.

[8 Marks]

(ii) Test $H_0 : \beta_1 = 0$ versus $H_a : \beta_1 < 0$ with $\alpha = 0.05$. Identify the corresponding attained significance level.

[8 Marks]

(iii) Find a 90% condence interval for the expected strength of concrete when the water/cement ratio is 1.5 Explain what would happen to the confidence interval if we computed the interval around the water/cement ratio is 2.7

[9 Marks]

END OF EXAMINATION PAPER