74

UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2005

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
		EACH QUESTION CARRIES 25 MARKS.
		MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) (i) Given P(- 3 , - 4 , - 5) in Cartesian coordinate system, find its cylindrical and spherical coordinates . (4 marks)

(ii) Given P(10 , 120^0 , 240^0) in spherical coordinate system, find its Cartesian and cylindrical coordinates . Express $\vec{e}_r = \vec{e}_x a_1 + \vec{e}_y a_2 + \vec{e}_z a_3$ and find the values of a_1 , a_2 *and* a_3 . (6 marks)

(b) Express $\vec{e}_r$, $\vec{e}_\theta$ *and* $\vec{e}_\phi$ in terms of $\vec{e}_x$, $\vec{e}_y$ *and* $\vec{e}_z$ and deduce that

$$\frac{d\,\vec{e}_\theta}{d\,t} = -\,\vec{e}_r \frac{d\,\theta}{d\,t} + \vec{e}_\phi \cos\theta \frac{d\,\phi}{d\,t} \quad .$$

(9 marks)

(c) Given $f = x^2 + y^2$, find the magnitude and direction of $\vec{\nabla} f$ at $x = 1 , y = 1$ *and* $z = 0$. Draw $f = 1$ and $f = 4$ two equal f surfaces on $z = 0$ plane (i.e., $x - y$ *plane*) and indicate on the diagram what should be the direction of $\vec{\nabla} f$ and also estimate the approximated magnitude of $\vec{\nabla} f$ at the given point $x = 1 , y = 1$ *and* $z = 0$ from your diagram . (6 marks)

Question two

(a) For the rectangular coordinate system, prove the following vector identity :

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) - \nabla^2 \vec{F}$$

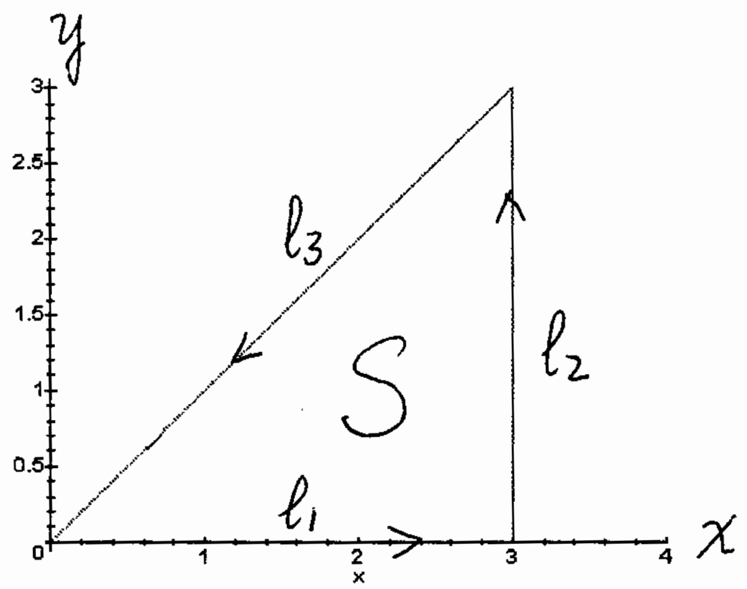
where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z$ and

$$\nabla^2 \vec{F} = \vec{e}_x \nabla^2 F_x + \vec{e}_y \nabla^2 F_y + \vec{e}_z \nabla^2 F_z$$ (10 marks)

(b) Given $\vec{F} = \vec{e}_x (x^2 + y^2) + \vec{e}_y (y^2 + z^2) + \vec{e}_z (z^2 + x^2)$

(i) evaluate the value of $\oint_l \vec{F} \bullet d\vec{l}$ where $l = l_1 + l_2 + l_3$ is on $z = 0$

(i.e., $x - y$ *plane*) and is shown below :



(8 marks)

Question two (continued)

$l_1 : y = 0 \ , \ x \ from \ 0 \ to \ 3$

$l_2 : x = 3 \ , \ y \ from \ 0 \ to \ 3$

$l_3 : y = x \ , \ x \ from \ 3 \ to \ 0$

(ii) Find $\vec{\nabla} \times \vec{F}$ and then evaluate the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by the given closed loop l in (i). (7 marks)

Question three

The following non-homogeneous differential equation represents a simple harmonic oscillator of mass $m = 2 \ kg$ and spring force constant $K = 26 \ \frac{N}{m}$ forced to oscillate in an viscous fluid :

$$2 \frac{d^2 x(t)}{d t^2} + 8 \frac{d x(t)}{d t} + 26 x(t) = f(t)$$

where $x(t)$: displacement from its resting position

$8 \frac{d x(t)}{d t}$: retardation force by the viscous fluid

$f(t)$: externally applied driving force

(a) Find and write down the general solution to the homogeneous part of the above given differential equation , i.e., $2 \frac{d^2 x(t)}{d t^2} + 8 \frac{d x(t)}{d t} + 26 x(t) = 0$ (5 marks)

(b) If the driving force is given as $f(t) = 8 \sin(5t)$, set the particular solution of the given non-homogeneous differential equation as $x(t) = k_1 \cos(5t) + k_2 \sin(5t)$ and find the values of k_1 *and* k_2 , (10 marks)

(c) (i) Combine the obtained solutions in (a) and (b) to write down the general solution of the given non-homogeneous differential equation , (2 marks)

(ii) If the given initial conditions for the system are $x(0) = 4$ and $\left. \frac{d x(t)}{d t} \right|_{t=0} = 0$, find the values of the arbitrary constants in (c)(i) and thus the specific solution for the given system. (8 marks)

Question four

(a) Given the three-dimensional Laplace equation in cylindrical coordinate system as

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f(\rho,\phi,z)}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f(\rho,\phi,z)}{\partial \phi^2} + \frac{\partial^2 f(\rho,\phi,z)}{\partial z^2} = 0$$

Setting $f(\rho,\phi,z) = F(\rho)\, G(\phi)\, H(z)$ and applying the technique of separation of variables , deduce three ordinary differential equations for $F(\rho)$, $G(\phi)$ *and* $H(z)$ from the given partial differential equation . (8 marks)

(b) Given the following differential equation $\dfrac{d^2 y(x)}{d x^2} + 4\, y(x) = 0$, using the power series method , i.e., set $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ with $a_0 \neq 0$ and substituting it back into the given differential equation ,

(i) requiring the coefficients of the two lowest power terms for x , i.e., x^{s-2} and x^{s-1} , to be zero and thus write down the indicial equations . From these equations deduce that $s = 0 , 1$ and $a_1 = 0$, (7 marks)

(ii) requiring the coefficients of all the rest power terms for x , i.e., x^{s+n} with $n = 0 , 1 , 2 , 3 , \cdots$, to be zero and deduce the recurrence relation ,

(4 marks)

(iii) for $s = 1$ and $a_1 = 0$, set $a_0 = 1$ and using the recurrence relation in (b) , find the values of a_2 , a_3 , a_4 , a_5 , a_6 and write down one of the independent solutions of the given differential equation up to $n = 6$ terms, i.e., $\sum_{n=0}^{6} a_n\, x^{n+s}$.

(6 marks)

Question five

(a) Given $m\frac{d^2x}{dt^2} = -kx$, and $m = 3\ kg$ & $k = 27\ \frac{N}{m}$

(i) find the values of the angular frequency , frequency and period of the given simple harmonic oscillator system , (3 marks)

(ii) write down the general solution of the given problem . (2 marks)

(b) Two simple harmonic oscillators (one is represented by m_1 *and* k_1 and the other represented by m_2 *and* k_2) are jointed together by a spring of spring constant K . The equations of motion for the system are :

$$\begin{cases} m_1\frac{d^2x_1(t)}{dt^2} = -(k_1+K)x_1(t)+Kx_2(t) \\ m_2\frac{d^2x_2(t)}{dt^2} = Kx_1(t)-(k_2+K)x_2(t) \end{cases}$$

where $x_1(t)$ and $x_2(t)$ are the displacement from their respective resting position .

If $m_1 = 1\ kg$, $m_2 = 3\ kg$, $k_1 = 3\ \frac{N}{m}$, $k_2 = 9\ \frac{N}{m}$ and $K = 9\ \frac{N}{m}$,

(i) show that the coupled differential equations for the system can be simplified to be :

$$\begin{cases} \frac{d^2x_1(t)}{dt^2} = -12\,x_1(t) + 9\,x_2(t) \\ \frac{d^2x_2(t)}{dt^2} = 3\,x_1(t) - 6\,x_2(t) \end{cases}$$

(2 marks)

Question five (continued)

(ii) setting $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$, and showing deduction details, find the eigenfrequencies ω of the given coupled system, (6 marks)

(iii) find the eigenvectors of the given coupled system corresponding to each eigenfrequencies found in (b)(ii), (6 marks)

(iv) find the normal coordinates of the given coupled system corresponding to each eigenfrequencies found in (b)(ii) . (6 marks)

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla}\, f = \vec{e}_1\,\frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2\,\frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3\,\frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1\,h_2\,h_3}\left(\frac{\partial(F_1\,h_2\,h_3)}{\partial u_1} + \frac{\partial(F_2\,h_1\,h_3)}{\partial u_2} + \frac{\partial(F_3\,h_1\,h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2h_3}\left(\frac{\partial(F_3h_3)}{\partial u_2} - \frac{\partial(F_2h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1h_3}\left(\frac{\partial(F_1h_1)}{\partial u_3} - \frac{\partial(F_3h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1h_2}\left(\frac{\partial(F_2h_2)}{\partial u_1} - \frac{\partial(F_1h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1\,F_1 + \vec{e}_2\,F_2 + \vec{e}_3\,F_3$ and

$(u_1\,,u_2\,,u_3)$ represents $(x\,,y\,,z)$ for rectangular coordinate system
represents $(\rho\,,\phi\,,z)$ for cylindrical coordinate system
represents $(r\,,\theta\,,\phi)$ for spherical coordinate system

$(\vec{e}_1\,,\vec{e}_2\,,\vec{e}_3)$ represents $(\vec{e}_x\,,\vec{e}_y\,,\vec{e}_z)$ for rectangular coordinate system
represents $(\vec{e}_\rho\,,\vec{e}_\phi\,,\vec{e}_z)$ for cylindrical coordinate system
represents $(\vec{e}_r\,,\vec{e}_\theta\,,\vec{e}_\phi)$ for spherical coordinate system

$(h_1\,,h_2\,,h_3)$ represents $(1\,,1\,,1)$ for rectangular coordinate system
represents $(1\,,\rho\,,1)$ for cylindrical coordinate system
represents $(1\,,r\,,r\sin\theta)$ for spherical coordinate system