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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2005

TITLE OF PAPER : MATHEMATICAL METHODS FOR PHYSICISTS

COURSE NUMBER : P272

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.

EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICISTS

Question one

(a) (i) Given the rectangular coordinates of a point P as $(-8\ ,\ 4\ ,\ -2)$, find its cylindrical and spherical coordinates respectively . Express the answers of angles in degrees . (4 marks)

(ii) Given the spherical coordinates of a point P as $(5\,,120^0\,,150^0)$, find its cylindrical and rectangular coordinates respectively . (4 marks)

(b) For a point P on $x-y\ plane$, i.e., $z=0$,

(i) draw the rectangular unit vectors $\vec{e}_x$, $\vec{e}_y$ as well as the cylindrical unit vectors $\vec{e}_\rho$, $\vec{e}_\phi$ for the given point on $x-y\ plane$, (3 marks)

(ii) express $\vec{e}_\rho$, $\vec{e}_\phi$ in terms of $\vec{e}_x$, $\vec{e}_y$ and deduce that

$d\,\vec{e}_\phi = -\,\vec{e}_\rho\, d\,\phi$ and $d\,\vec{e}_\rho = \vec{e}_\phi\, d\,\phi$ (5 marks)

(c) Given $g(r,\theta,\phi)=\dfrac{10}{r^2}+r\sin\theta\cos\phi$,

(i) find $\vec{\nabla}\, g$, (3 marks)

(ii) evaluate $\vec{\nabla}\, g$ at the point $P:(5\,,150^0\,,30^0)$ and also find the directional derivative of g along the direction of $\vec{e}_r\,3+\vec{e}_\theta\,4$. (6 marks)

Question two

(a) Given any scalar function f and any vector function $\vec{F}$ in Cartesian coordinate system, (i.e., $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z$ where F_x , F_y , F_z *and* f are all functions of (x, y, z)),verify the following identity :

$$\vec{\nabla} \cdot (f \vec{F}) \equiv \vec{F} \cdot (\vec{\nabla} f) + f (\vec{\nabla} \cdot \vec{F})$$ (8 marks)

(b) Given a vector field $\vec{G}(\rho, \phi, z) = \vec{e}_\rho \rho^2 + \vec{e}_\phi \rho z + \vec{e}_z (\rho^2 + \rho z)$,

(i) carry out the following closed surface integration of $\oiint_S \vec{G} \cdot d\vec{s}$

where S : the surface enclose the whole of a cylindrical tube of radius ρ_0 and height h , with $z - axis$ coincides with the axial line of the tube , i.e.,

$S = S_1 + S_2 + S_3$ where

S_1 : circular disk surface of radius ρ_0 on $z = 0$ plane

S_2 : circular disk surface of radius ρ_0 on $z = h$ plane

S_3 : circular tube surface of radius ρ_0 on $\rho = \rho_0$ plane with height h

Express your answer in terms of ρ_0 and h . (10 marks)

(ii) carry out the value integral of $\iiint_V (\vec{\nabla} \cdot \vec{G})\, dv$ where V : the volume of the given cylindrical tube , i.e., the volume enclosed by the closed surface S specified in (b)(i) . Compare it with that obtained in (b)(i) and make brief comments.

(7 marks)

Question three

If the transverse wave amplitude function $u(x,t)$ of a certain vibrating string follows the following partial differential equation : $\dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{1}{4}\dfrac{\partial^2 u(x,t)}{\partial t^2} = 0$,

(a) set $u(x,t) = X(x)\,T(t)$ and utilize the separation variable scheme to deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 X(x)}{d x^2} = -\,k^2\, X(x) \\ \dfrac{d^2 T(t)}{d t^2} = -\,4\,k^2\, T(t) \end{cases}$$

where k is a separation constant , (4 marks)

(b) (i) by direct substitution, show that $X(x) = A_k \cos(k\,x) + B_k \sin(k\,x)$ and $T(t) = C_k \cos(2\,k\,t) + D_k \sin(2\,k\,t)$ are a general solution to the ordinary differential equations in (a) with A_k , B_k , C_k *and* D_k as arbitrary constants,

(3 marks)

(ii) given the length of the vibrating string as three metres with both ends fixed , i.e., $u(0,t) = 0 = u(3,t)$, find the eigenvalues of k and write down the general solution of $u(x,t)$ to include all the eigenvalues of k , (6 marks)

(c) given the initial condition as $\left.\dfrac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$ and

$$u(x,0) = \begin{cases} x & for\ 0 \le x \le \dfrac{3}{2} \\ -x+3 & for\ \dfrac{3}{2} \le x \le 3 \end{cases}$$

, determine the specific values of those arbitrary constants in the general solution of $u(x,t)$ written down in (b)(ii) and thus write down the specific solution of this given problem.

(hint : $\displaystyle\int_0^3 \sin(\frac{n\pi}{3}x)\ \sin(\frac{m\pi}{3}x)\ dx = \begin{cases} \dfrac{3}{2} & if\ \ n = m \\ 0 & if\ \ n \ne m \end{cases}$

where $n\ and\ m$ are non-zero positive integers) (12 marks)

Question four

Given the following differential equation $\left(1-x^2\right)\dfrac{d^2 y(x)}{dx^2}+3x\dfrac{d\,y(x)}{dx}+4\,y(x)=0$,

using the power series method , i.e., set $y(x)=\sum_{n=0}^{\infty} a_n\, x^{n+s}$ with $a_0 \neq 0$ and substituting it

back to the given differential equation ,

(a) requiring the coefficients of the lowest power terms for x , i.e., x^{s-2} and x^{s-1},

to be zero and thus write down the indicial equations. From these equations

find the values of s (possibly also the values of a_1), (6 marks)

(b) requiring the coefficients of all the rest power terms for x , i.e., x^{s+n} with

$n=0,1,2,3,\cdots$, to be zero and find the recurrence relation, (5 marks)

(c) (i) using the recurrence relation in (b) , find the values of a_2 , a_3 ,$\cdots$ a_6 if $a_0 = 1$

for each value of s found in (a) . (12 marks)

(ii) write down the general solution of the given differential equation. (2 marks)

Question five

(a) Given $m \frac{d^2 x}{dt^2} = -kx$, and $m = \frac{1}{2}\ kg$ & $k = 8\ \frac{N}{m}$

(i) find the values of the angular frequency , frequency and period of the given simple harmonic oscillator system , (3 marks)

(ii) write down the general solution of the given problem . (2 marks)

(b) Two simple harmonic oscillators (one is represented by m_1 and k_1 and the other represented by m_2 and k_2) are jointed together by a spring of spring constant k_3.

The coupled differential equations are simplified to be :

$$\begin{cases} \frac{d^2 x_1}{dt^2} = -9x_1 + 4x_2 \\ \frac{d^2 x_2}{dt^2} = 6x_1 - 4x_2 \end{cases}$$

(i) set $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$, deduce the following matrix equation $\lambda X = AX$

where $\lambda = -\omega^2$, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $A = \begin{pmatrix} -9 & 4 \\ 6 & -4 \end{pmatrix}$

(4 marks)

(ii) find the eigenfrequencies ω for the matrix equation in (b)(i) (6 marks)

(iii) find the eigenvectors corresponding to the eigenfrequencies found in (b)(ii) respectively , (4 marks)

(iv) find the normal coordinates of the system . (6 marks)

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3) represents (x, y, z) for rectangular coordinate system

represents (ρ, ϕ, z) for cylindrical coordinate system

represents (r, θ, ϕ) for spherical coordinate system

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ represents $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ for rectangular coordinate system

represents $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$ for cylindrical coordinate system

represents $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ for spherical coordinate system

(h_1, h_2, h_3) represents $(1, 1, 1)$ for rectangular coordinate system

represents $(1, \rho, 1)$ for cylindrical coordinate system

represents $(1, r, r\sin\theta)$ for spherical coordinate system