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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2005

TITLE OF PAPER : ELECTROMAGNETIC THEORY

COURSE NUMBER : P331

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS FOURTEEN PAGES, INCLUDING THIS PAGE.

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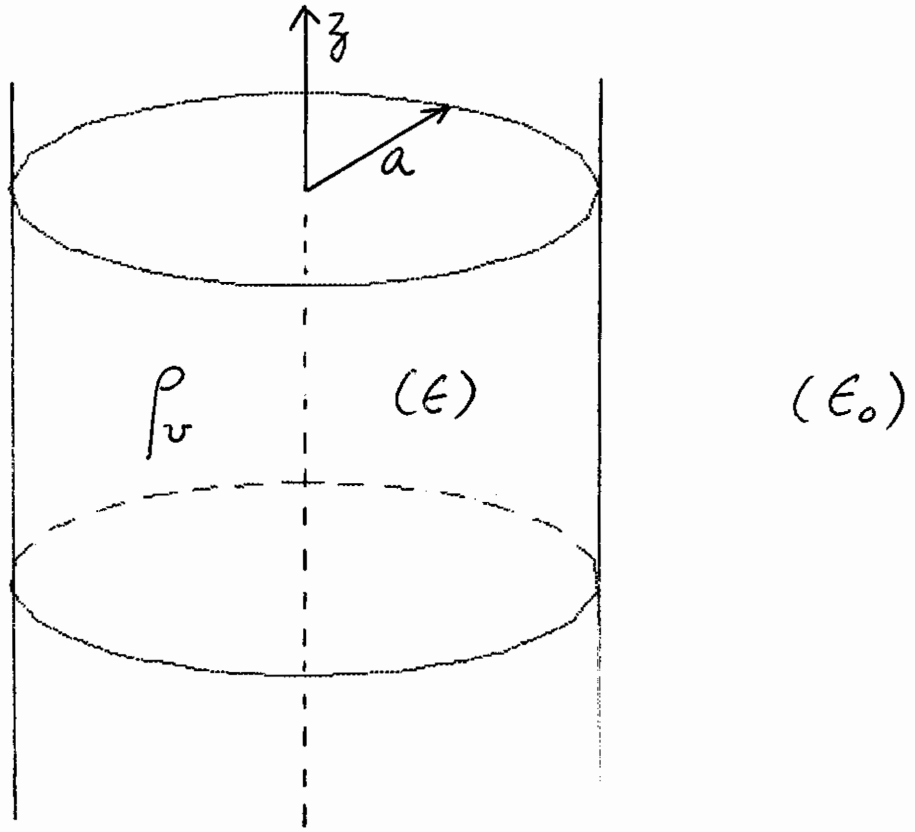
P331 Electromagnetic Theory I

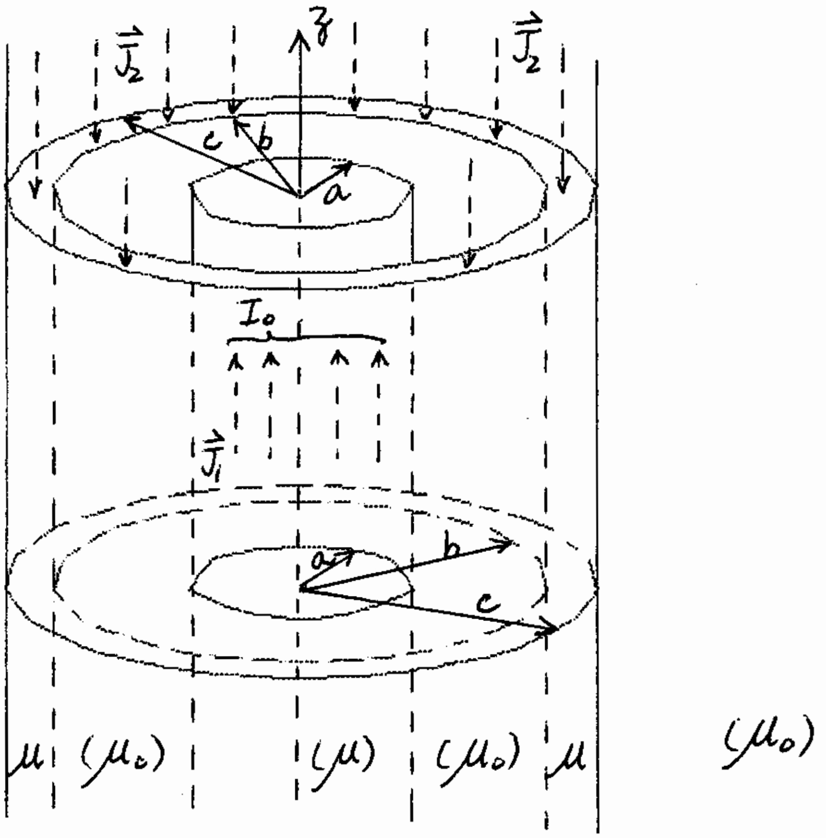
Question one

(a) For a very long wire of cross sectional radius a , if uniformly charged , i.e., the volume charge density ρ_v = constant for $0 \leq \rho \leq a$, as shown below ,



use the integral form of Coulomb's Law , i.e., $\oint\!\!\oint_S \vec{D} \bullet d\vec{s} = \iiint_V \rho_v \, dv$, to find the electric field $\vec{E}$ in the $0 \leq \rho \leq a$ (material region with permitivity ε) and $\rho \geq a$ (free space with permitivity ε_0) regions respectively, (8 marks)

(b) A very long straight coaxial cable, with an inner solid wire of radius a carries an uniform current density $\vec{J}_1 = \vec{a}_z \dfrac{I_0}{\pi\, a^2}$ and outer hollow wire with inner radius b and outer radius c carries an uniform current density $\vec{J}_2 = \vec{a}_z \dfrac{-\,I_0}{\pi\,(c^2 - b^2)}$ where I_0 is the total current in the inner wire , as shown in the diagram below :



(i) Use the integral form of Ampere's Law , i.e., $\oint_l \vec{H} \bullet d\vec{l} = \iint_S \vec{J} \bullet d\vec{s}$, to find the magnetic field $\vec{B}$ for $0 \le \rho \le a$ (material region with permeability μ), $a \le \rho \le b$ (free space with permeability μ_0), $b \le \rho \le c$ (material region with permeability μ) and $\rho \ge c$ (free space with permeability μ_0) regions respectively , (12 marks)

Question one (continued)

(ii) Find the total magnetic energy , i.e., $U_m = \frac{1}{2}\iiint_V \left(\vec{B} \bullet \vec{H}\right) d\,v$, stored in the given cable of length l between the wires , i.e., $a \le \rho \le b$, and then find the external self-inductance L_e for this length of coaxial cable. (5 marks)

(Note : $U_m = \frac{1}{2} L_e\, I_0^2$ and $\iiint_V d\,v = \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \int_{z=0}^{l} \rho\, d\rho\, d\phi\, d z$)

Question two

(a) Write down the continuity equation of electric charge in its integral and differential forms.

Show that without introducing the displacement current term ,i.e., $\frac{\partial \vec{D}}{\partial t}$, in the equation for Ampere's law , i.e., $\vec{\nabla} \times \vec{H} = \vec{J}$ instead of $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, Maxwell's equations would be in contradiction with the continuity equation . (8 marks)

(b) In a conductive region, based on Drude model, the force on a conducting electron by $\vec{E}$ is $-e\vec{E}$ and the average momentum change caused by the ion lattice of the conductor is $-\frac{m_e \vec{v}_d}{\tau_c}$ and , thus , in a steady state situation one has $-e\vec{E} - \frac{m_e \vec{v}_d}{\tau_c} = 0$,

(i) Explain briefly $\vec{v}_d$ and τ_c . (2 marks)

(ii) Using the current density in the conductive region $\vec{J} = \rho_v \vec{v}_d = -n e \vec{v}_d$, deduce the following point form of Ohm's Law $\vec{J} = \sigma \vec{E}$ where $\sigma = \frac{n e^2}{m_e} \tau_c$ for the dc case . (3 marks)

(iii) Sodium has an atomic density of $2.3 \times 10^{28} \frac{atom}{m^3}$. at room temperature , and with one outer-orbit electron available for conduction, find the value of τ_c for sodium if its dc conductivity $\sigma = 2.1 \times 10^7 \frac{1}{m\,\Omega}$ at room temperature .

(3 marks)

(c) For the time-independent (or static) case , i.e., $\frac{\partial}{\partial t} \rightarrow 0$, setting $\vec{E} = -\vec{\nabla}\Phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, where Φ and $\vec{A}$ are the scalar and vector potentials respectively ,

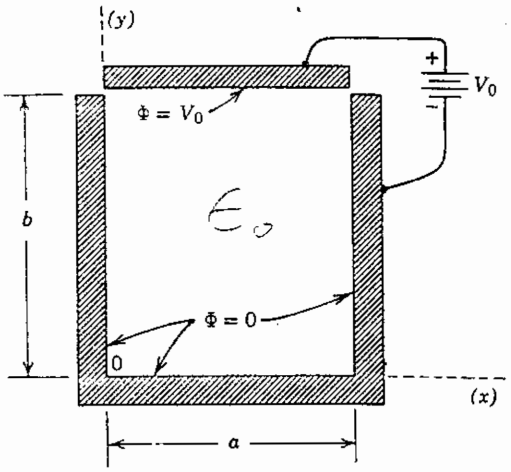
(i) show that Maxwell's equations for Faraday's law and for the magnetic Coulomb's law are automatically satisfied . (3 marks)

(ii) Use Coulomb's gauge ,i.e., $\vec{\nabla} \bullet \vec{A} = 0$, to deduce the following equations :

$\nabla^2 \vec{A} = -\mu \vec{J}$ and $\nabla^2 \Phi = -\frac{\rho_v}{\varepsilon}$. (6 marks)

Question three

An infinitely long, rectangular U shaped conducting channel is insulated at the corners from a conducting plate forming the fourth side with interior dimensions as shown below :



The differential equation for the electric potential $\Phi(x,y)$ in between the two conductors is

$$\frac{\partial^2 \Phi(x,y)}{\partial x^2} + \frac{\partial^2 \Phi(x,y)}{\partial y^2} = 0$$

(a) Setting $\Phi(x,y) = X(x)\,Y(y)$ and applying the technique of separation of variables , deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 X(x)}{d x^2} = -k^2 X(x) \cdots\cdots (1) \\ \dfrac{d^2 Y(y)}{d y^2} = k^2 Y(y) \cdots\cdots (2) \end{cases}$$

where $k\,(>0)$ is a separation constant . (3 marks)

(b) For a particular value of k, show that the general solution for equations (1) & (2) can be written as : $X_k(x) = (A_k \cos(kx) + B_k \sin(kx))$,

$Y_k(y) = (C_k \cosh(ky) + D_k \sinh(ky))$ and $\Phi_k(x,y) = X_k(x)\, Y_k(y)$

where A_k , B_k , C_k and D_k are arbitrary constants. (3 marks)

(c) Applying the given boundary conditions for Φ at $x = 0$, $y = 0$ and $x = a$,

i.e., $\Phi_k(0,y) = \Phi_k(a,y) = \Phi_k(x,0) = 0$, deduce that $A_k = 0$,

$C_k = 0$ and $k = \dfrac{n\pi}{a}$, $n = 1,2,\cdots$. (6 marks)

(d) The general solution now can be written as $\Phi(x,y) = \sum_{n=1}^{\infty} \Phi_n(x,y)$ where

$\Phi_n(x,y) = E_n \sinh(\frac{n\pi}{a} y) \sin(\frac{n\pi}{a} x)$, (k re-indexed as n and $B_n D_n \equiv E_n$)

Apply the last boundary condition at $y = b$, i.e., $\Phi(x,b) = V_0$, utilize the Fourier

integral $\int_0^a \sin(\frac{n\pi}{a} x) \sin(\frac{m\pi}{a} x)\, dx = \begin{cases} 0 & if \;\; n \neq m \\ \frac{a}{2} & if \;\; n = m \end{cases}$ to find

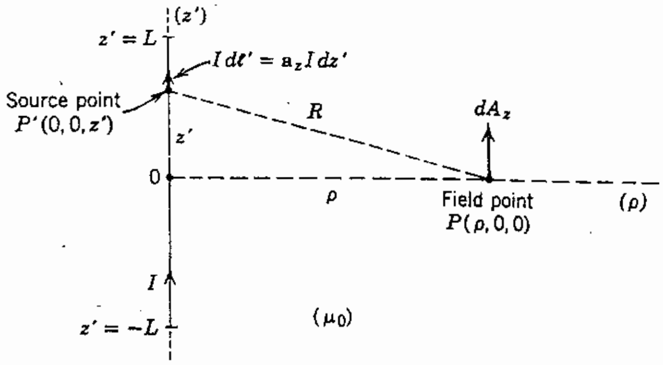
E_n $n = 1,2,3,\cdots$. Show that the specific solution for this problem is

$$\Phi(x,y) = \frac{4V_0}{\pi}\left(\frac{\sinh(\frac{\pi y}{a})}{\sinh(\frac{\pi b}{a})} \sin(\frac{\pi x}{a}) + \frac{1}{3}\frac{\sinh(\frac{3\pi y}{a})}{\sinh(\frac{3\pi b}{a})} \sin(\frac{3\pi x}{a}) + \cdots \right)$$

(13 marks)

Question four

(a) A straight thin wire of finite length $2L$ in free space carries a direct current I as shown in the diagram below :



(i) For a field point $P(\rho,0,0)$ on a plane bisecting the given conducting wire , use $\vec{A}(\rho,0,0) = \int_l \frac{\mu_0 \, I \, d\vec{l}\,'}{4\pi R}$ where $d\vec{l}\,' = \vec{a}_z \, dz'$, $z' = -L \;\; to \;\; L$ and $R = \sqrt{\rho^2 + (z')^2}$, to deduce that

$$\vec{A} = \vec{a}_z \frac{\mu_0 \, I}{4\pi} \ln\left(\frac{\sqrt{L^2 + \rho^2} + L}{\sqrt{L^2 + \rho^2} - L}\right) \qquad \text{(6 marks)}$$

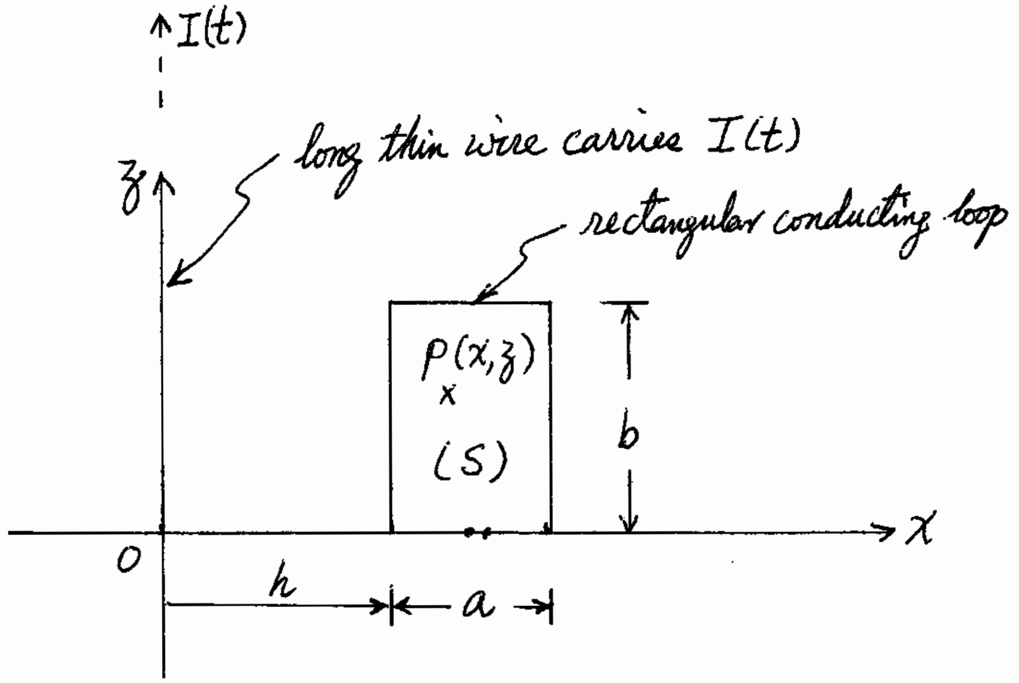
(Hint : $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right)$)

(ii) Find the magnetic field $\vec{B}$ from the vector potential $\vec{A}$ obtained in (a)(i).

Deduce that $\vec{B} = \vec{a}_\phi \frac{\mu_0 \, I}{2\pi\rho} \frac{L}{\sqrt{L^2 + \rho^2}}$ and thus show that if $L >> \rho$ then $\vec{B} \approx \vec{a}_\phi \frac{\mu_0 \, I}{2\pi\rho}$ which is the $\vec{B}$ field generated by an infinitely long current carrying wire. (7 marks)

Question four (continued)

(b) A rectangular conducting loop is situated near a very long thin straight wire carrying a slow time-variant current $I(t)$ as show in the diagram below :



(i) Use the quasi-static expression of $\vec{B}$ in (a)(ii) for an infinitely long current carrying wire with I replaced by $I(t)$, $\vec{a}_\phi$ replaced by $\vec{a}_y$ and ρ replaced by x , i.e., $\vec{B} = \vec{a}_y \dfrac{\mu_0 I(t)}{2\pi x}$, to find the total $\vec{B}$ flux lines out of the given rectangular conducting loop , i.e., $\iint_S \vec{B} \bullet d\vec{s}$ where $d\vec{s} = -\vec{a}_y\, dx\, dz$, with $x = h$ *to* $h + a$ and $z = 0$ *to* b .

(8 marks)

(ii) Use Faraday's Law, find the induced $e.m.f.$ on the given rectangular conducting loop if $I(t) = I_0 \cos(\omega t)$. (4 marks)

Question five

(a) (i) If $\vec{E}$ and $\vec{H}$ are replaced by $\vec{\hat{E}}(\vec{r})e^{i\omega t}$ and $\vec{\hat{H}}(\vec{r})e^{i\omega t}$ respectively, deduce the following time harmonic Maxwell's equations for a material region represented by parameters of μ , ε , σ and also having $\rho_v = 0$ as :

$$\vec{\nabla} \cdot \vec{\hat{E}}(\vec{r}) = 0 \quad , \quad \vec{\nabla} \cdot \vec{\hat{H}}(\vec{r}) = 0 \quad , \quad \vec{\nabla} \times \vec{\hat{E}}(\vec{r}) = -\, i\omega\,\mu\,\vec{\hat{H}}(\vec{r}) \quad \text{and}$$

$$\vec{\nabla} \times \vec{\hat{H}}(\vec{r}) = \left(\sigma + i\omega\,\varepsilon\right)\vec{\hat{E}}(\vec{r})$$

(4 marks)

(ii) If further assuming that $\vec{\hat{E}}$ and $\vec{\hat{H}}$ are functions of z only, i.e.,

$$\vec{\hat{E}}(\vec{r}) = \vec{e}_x\,\hat{E}_x(z) + \vec{e}_y\,\hat{E}_y(z) + \vec{e}_z\,\hat{E}_z(z) \quad \text{and}$$

$$\vec{\hat{H}}(\vec{r}) = \vec{e}_x\,\hat{H}_x(z) + \vec{e}_y\,\hat{H}_y(z) + \vec{e}_z\,\hat{H}_z(z) \ ,$$

deduce the following equations :

$$\hat{E}_z(z) = 0 = \hat{H}_z(z) \quad , \quad \frac{d\,\hat{E}_x(z)}{d\,z} = -\, i\omega\,\mu\,\hat{H}_y(z) \quad ,$$

$$\frac{d\,\hat{E}_y(z)}{d\,z} = i\omega\,\mu\,\hat{H}_x(z) \quad , \quad \frac{d\,\hat{H}_x(z)}{d\,z} = (\sigma + i\omega\,\varepsilon)\,\hat{E}_y(z) \quad ,$$

$$\frac{d\,\hat{H}_y(z)}{d\,z} = -\,(\sigma + i\omega\,\varepsilon)\,\hat{E}_x(z) \quad \text{and}$$

$$\frac{d^2\,\hat{E}_x(z)}{d\,z^2} + (\omega^2\,\mu\,\varepsilon - i\omega\,\mu\,\sigma)\,\hat{E}_x(z) = 0$$

(8 marks)

(b) An uniform plane wave travelling along $+z$ direction with the field components

$E_x(z)$ and $H_y(z)$ has an complex electric field amplitude of $100\, e^{i\,30^\circ}\, \dfrac{V}{m}$ and

propagates at $f = 10^6\ Hz$ in a material region having the parameters $\mu = \mu_0$,

$\varepsilon = 9\,\varepsilon_0$ and $\dfrac{\sigma}{\omega\,\varepsilon} = 0.5$.

(i) Find the values of the propagation constant $\hat{\gamma}$ $(= \alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave . (4 marks)

(ii) Express the electric and magnetic fields in both their complex and real-time forms, with the numerical values of (b)(i) inserted . (4 marks)

(iii) Find the values of the penetration depth , wave length and phase velocity of the given wave. (5 marks)

USEFUL INFORMATION

$e = 1.6 \times 10^{-19} \quad C \quad , \quad m_e = 9.1 \times 10^{-31} \quad kg$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3 \times 10^{8}\ \frac{m}{s} \quad , \quad \beta_0 \equiv \omega\,\sqrt{\mu_0\,\varepsilon_0} \quad , \quad \mu_0 = 4\pi \times 10^{-7}\ \frac{H}{m}$$

$$, \varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m} \quad , \quad \hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1 + \left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$$\eta_0 \equiv \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \quad \Omega = 377 \quad \Omega$$

$$\alpha = \frac{\omega\,\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1} \quad , \quad \beta = \frac{\omega\,\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1} \quad ,$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon} \quad , \quad \vec{\nabla}\cdot\vec{B} = 0 \quad , \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \vec{\nabla}\times\vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\frac{\partial\vec{E}}{\partial t} \quad ,$$

$$\vec{\nabla}\times\vec{B} = \mu\,\sigma\,\vec{E} + \mu\,\varepsilon\,\frac{\partial\vec{E}}{\partial t} \,,$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla}\cdot\vec{F}\right) - \nabla^2\vec{F} \quad , \quad \oiint_S \vec{F}\cdot d\vec{s} = \iiint_V \vec{\nabla}\cdot\vec{F}\; dv \quad ,$$

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{F}\right) \equiv 0 \quad , \quad \vec{\nabla}\times\left(\vec{\nabla}f\right) \equiv 0 \quad ,$$

$$\vec{\nabla}f = \vec{e}_r\,\frac{\partial f}{\partial r} + \vec{e}_\theta\,\frac{1}{r}\frac{\partial f}{\partial\theta} + \vec{e}_\phi\,\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} \quad ,$$

$$\vec{\nabla}\bullet\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

USEFUL INFORMATION (continued)

$$\vec{\nabla}\cdot\vec{F}=\frac{1}{r^2}\frac{\partial\left(r^2 F_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta\, F_\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\quad,$$

$$\vec{\nabla}\times\vec{F}=\vec{e}_x\left[\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right]+\vec{e}_y\left[\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right]+\vec{e}_z\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right]\quad,$$

$$\vec{\nabla}\times\vec{F}=\frac{\vec{e}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(F_\phi\sin\theta\right)-\frac{\partial F_\theta}{\partial\phi}\right]+\frac{\vec{e}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{\partial\left(r F_\phi\right)}{\partial r}\right]$$
$$+\frac{\vec{e}_\phi}{r}\left[\frac{\partial\left(r F_\theta\right)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right]\quad,$$

$$\vec{\nabla}\times\vec{F}=\vec{a}_\rho\left[\frac{1}{\rho}\frac{\partial F_z}{\partial\phi}-\frac{\partial F_\phi}{\partial z}\right]+\vec{a}_\phi\left[\frac{\partial F_\rho}{\partial z}-\frac{\partial F_z}{\partial\rho}\right]$$
$$+\vec{a}_z\left[\frac{1}{\rho}\frac{\partial(\rho F_\phi)}{\partial\rho}-\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi}\right]$$

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$