124

UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS & ELECTRONIC ENGINEERING

MAIN EXAMINATION 2005

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.

EACH QUESTION CARRIES **25** MARKS.
MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **FIVE** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1.

(a) (i) A typical thermal neutron kinetic energy equals $\frac{3}{2}kT$ at *T=300K*. [5]

What is its velocity and its de Broglie wavelength?
In a specimen where inter-atomic distances are of the order $10^{-10}m$, will there be diffraction of neutrons.
(ii) The average lifetime of an excited state of an atom is about 10^{-8} sec. Using this as Δt for the emission of a photon, compute the minimum $\Delta \nu$ permitted by the uncertainty principle. What fraction of ν is this if the wavelength of the spectral line involved is 4.0×10^{-7} m ? [5]

Given: $\hbar = 1.0546 \times 10^{-34}$ *J s* , *c = velocity of light* $= 2.99792 \times 10^{8}$ $m\,s^{-1}$
mass of neutron $= 1.6749 \times 10^{-27}$ *kg* , $k = 1.3807 \times 10^{-23}\ JK^{-1}$.

(b) Explain: [10]
(i) What do you understand by stationary states.
(ii) The degenerate states.
(iii) Parity.
(iv) Complete set of states.
(v) Ortho-normal states and its physical significance.

(c) The wave function of a particle moving in one dimension is given by: [5]

$$\psi(x) = 0 \quad \text{for } x < 0$$
$$= B\sqrt{x}\exp(-\beta x) \quad \text{for } x \ge 0$$

where β is a real and positive constant.

Calculate the normalization constant B. (It is a function of β.)

Note: $\Gamma(z) = k^2 \int_0^\infty t^{z-1} \exp(-kt)\,dt \qquad \text{Re}\, z > 0, \text{Re}\, k > 0.$

$$\Gamma(n+1) = n! \quad \text{for } n = 1,2,\ldots \text{ and } \Gamma(1) = 1.$$

Q.2.
(i) Write down the Schrodinger equation for the potential given by [4]

$$V(x) = -V_0 \quad \text{if } |x| \le l, \text{ and } V_0 > 0,$$
$$= 0 \quad \text{if } |x| > l,$$

(ii) Solve the Schrodinger equation to determine the odd [*i.e.* $\psi(-x) = -\psi(x)$] bound energy eigenstates. [12]
(iii) Write down the normalization integral for the above energy states. [4]
Note: Do not attempt to evaluate the integral.

(iv) What do you get in the limit $V_0 \to \infty$ while keeping the energy [5]
$E' = E + V_0$ finite?

Q.3.
Following wave functions

(a) $u_0(x,y) = A\exp\left[-\frac{\alpha^2(x^2+y^2)}{2}\right]$

(b) $u_1(x,y) = B\,x\,y\,\exp\left[-\frac{\alpha^2(x^2+y^2)}{2}\right]$

(c) $u_2(x,y) = C\,(2\alpha^2y^2-1)\exp\left[-\frac{\alpha^2(x^2+y^2)}{2}\right]$

are the solutions of the eigenvalue problem $H u_n(x,y) = E_n u_n(x,y)$

with $H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{\hbar^2\alpha^4}{2m}(x^2+y^2)$.

(i) What is the expression for the potential? [2]
(ii) Determine E_n for each of them. [9]
(iii) What is the parity of each state? [3]
(iv) Determine the normalization constants A , B. [6]
(v) Are there any degenerate states among the three? If so, why? [5]

Q.4.
(a) show that

(i)) $[f(\vec{r}),p_x] = i\hbar\frac{\partial}{\partial x}f(\vec{r})$ where $\vec{r} = \hat{i}\,x+\hat{j}\,y+\hat{k}\,z$. [3]

(ii) $[x, p_x^3] = 3i\hbar p_x^2$ [5]
(iii) $[L_+, L_-] = 2\hbar L_z$ [5]
where $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$

(b) The Hamiltonian of a system with moment of inertia β is given by the expression

$$H = \frac{1}{2\beta}(L_x^2 + L_y^2)$$

Here L is orbital angular momentum.
(i) Show that $[L^2, L_i]=0$ for $i=x,y,z$. [3]
(ii) Show that $[H, L^2]=0$ and $[H, L_z]=0$. [4]
(iii) Show that the functions $Y_\ell^m(\vartheta,\varphi)$ are eigenfunctions of H .
Find an expression for the eigenvalue of the Hamiltonian. [5]

Q.5. The stationary Schrödinger equation for a particle moving in a central potential $V(r)$ is

$$E\Phi(r,\vartheta,\varphi) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}\right)+\frac{1}{2mr^2}\hat{L}^2\Phi(r,\vartheta,\varphi)+V(r)\Phi(r,\vartheta,\varphi),$$

where $\hat{L}$ is the angular momentum operator for the particle's motion.

(a) Write the wave function $\Phi(r,\theta,\phi)$ as a product of a radical function R (r) and an angular momentum eigenfunctions $Y_\ell^m(\vartheta,\varphi)$, [15] and derive the differential equation for *R(r)*. State the boundary conditions on *R(r)*.

(b) An electron in the Coulomb field of a proton with Hamiltonian H is in a state described the wave function

$$\phi = \frac{1}{5}\left[4\,\psi_{100} + 3\,\psi_{211}\right]$$

(i) What is the expectation value of the energy? [5]

(ii) What is the expectation value of L^2 and L_z ? [5]

Note that $\psi_{nlm} = R_{nl}(r)\,Y_l^m(\vartheta,\varphi)$ is the eigenfunction of H with energy $\frac{E_0}{n^2}$ where E_0 is a constant and

n = principal quantum number,
l = angular momentum quantum number
m = projection of angular momentum.

and $\int \psi^*_{n'l'm'}\,\psi_{nlm}\,d\tau = \delta_{n'n}\,\delta_{l'l}\,\delta_{m'm}$

@@@@END OF EXAMINATION@@@@

APPENDIX:

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

$[\, r_i , p_j \,] = i\hbar\, \delta_{ij}$ where $r_i = (\, x , y , z \,)$ and $p_i = (\, p_x , p_y , p_z \,)$,
$[\, L_x , L_y \,] = i\hbar\, L_z$, $[\, L_y , L_z \,] = i\hbar\, L_x$, $[\, L_z , L_x \,] = i\hbar\, L_y$ where $\vec{L} = \vec{r} \times \vec{p}$,

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2\, Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2\, Y_\ell^m(\vartheta,\varphi)$$
$$L_z\, Y_\ell^m(\vartheta,\varphi) = m\hbar\, Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_0^\infty exp(-t^2)\,dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1}\, exp(-at^2)\,dt = \frac{n\,!}{2\,a^{n+1}}$$ with Re a > 0, n =0,1,2,..

$$\int_0^\infty t^{2n}\, exp(-at^2)\,dt = \frac{1.3.5......(2n-1)}{2^{n+1}\,a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....

$$\int \sin^2(x)dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int \sin(mx)\sin(nx)dx = \frac{1}{2}\left[\frac{\sin\{(m-n)x\}}{(m-n)} - \frac{\sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int \sin(mx)\cos(nx)dx = -\frac{1}{2}\left[\frac{\cos[(m-n)x]}{(m-n)} + \frac{\cos[(m+n)x]}{(m+n)}\right]$$

$\int_{-\infty}^{\infty} H_n(\xi)\, H_m(\xi)\exp(-\xi^2)\,d\xi = \pi^{\frac{1}{2}}\, 2^n\, n!\, \delta_{nm}$ where $H(\xi)$ are Hermite polynomials and are real.