129

UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS & ELECTRONIC ENGINEERING

SUPPLEMENTARY EXAMINATION 2005

TITLE OF THE PAPER: **QUANTUM MECHANICS-I**

COURSE NUMBER : **P342**

TIME ALLOWED : **THREE HOURS**

INSTRUCTIONS:

ANSWER ANY **FOUR** OUT OF FIVE QUESTIONS.

EACH QUESTION CARRIES **25** MARKS.
MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **SIX** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1:

a) What is the deBroglie wavelength of alpha particle with kinetic energy of 7.7 MeV? In Rutherford's experiments, distances of the order of 10^{-13} m were involved, yet the analysis of the experiment did not include the wave properties of alpha-particle. Was this justified? [6 marks]

b) Electromagnetic radiation consists a collection of quanta known as photons with energy $h\nu$. Calculate the number of photons emitted by 100 watt source with $\lambda = 600 \times 10^{-9}$ m. [6 marks]

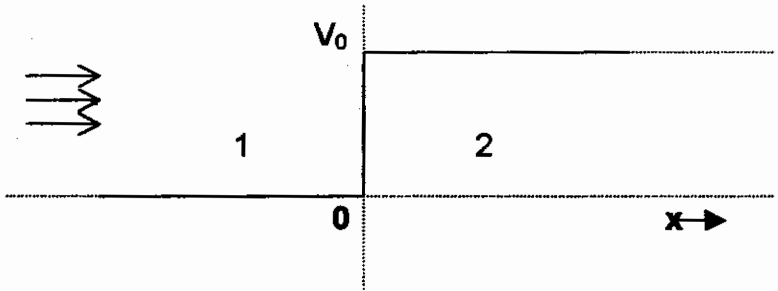
c) Explain the circumstance under which a system can be described by stationary states. [5 marks]

d) Write short notes on: [8 marks]

   (i) Ortho-normality
   (ii) Degenerate states
   (iii) Parity
   (iv) Complete set.

Q.2: Consider the step potential

$$V(x) = V_0 \quad x > 0$$
$$= 0 \quad x<0$$



Consider a current of particles of mass *m* propagating from left to right of energy $E > V_0$.

Define $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$, $k_2 = \sqrt{\frac{2m(E-V)}{\hbar^2}}$

Then the general solutions for the regions *1 (x<0)* and *2(x >0)* are

$$\phi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \quad , \quad \phi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$$

(i) State the boundary conditions on the solutions. [2]

(ii) Show that $B_2 = 0$ and $A_1 + B_1 = A_2$. [3]

(iii) Show that $\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$ *and* $\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$. [8]

(iv) Show that the probability current density

$$j(x) = \frac{\hbar}{2mi}\left[\phi^*(x)\frac{\partial\phi(x)}{\partial x} - \phi(x)\frac{\partial\phi^*(x)}{\partial x}\right] = \frac{\hbar k_2}{m}|A_2|^2 \quad [4]$$

(v) Using the definition of the reflection co-efficient R, show that

$$R = 1 - \frac{4k_1 k_2}{(k_1 + k_2)^2}$$ [8]

Q.3. (a) A particle is described by the wave function

$$\psi(x) = \left(\frac{\pi}{a}\right)^{-1/4} \exp(-a x^2 / 2)$$

Show that $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{2a}}$. [10]

(b) Consider a one dimensional physical system described by the Hamiltonian

$$H = \frac{p^2}{2m} + V(x)$$

(i) Show that $[H, x] = -\frac{i\hbar}{m} p$. [5]

(ii) For stationary state find $\langle p \rangle$. [5]

Note: For stationary states we have $H|\psi\rangle = \lambda|\psi\rangle$ where λ is the eigen value.

(c) The stationary states of a particle are given by the wave function

$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin(\frac{\pi n x}{L})$ and the corresponding energies are $E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}$

where n = 1 , 2 , 3 , ...

At time t =0 , the particle is in a state given by

$$\psi(x, t=0) = \frac{1}{\sqrt{2}}[\varphi_1(x) + \varphi_2(x)] .$$

Find the time dependent $\psi(x,t)$. [5]

Q.4. (a) The motion of a point mass m is described by a one dimensional potential

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

The orthonormal oscillator eigenfunctions are

$$\psi_n(x) = 2^{-n/2} (n!)^{-1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} x^2\right) H_n\left(\sqrt{\frac{mw}{\hbar}}\, x\right)$$

where H_n are Hermite polynomials and $n = 0, 1, 2, ...$

Find value of the following integrals:

$$\int_{-\infty}^{\infty} \psi_m^* x \psi_n \, dx \quad and \quad \int_{-\infty}^{\infty} \psi_m^* p_x \psi_n \, dx$$

for (i) $m=0$ and $n=1$ [5]

(ii) $m=1$ and $n=2$ [5]

(iii) $m=n$ for $n=0,1,$ and 2. [5]

(b) The motion of a point mass m is described by a two dimensional potential

$$V(x,y)=\frac{1}{2}m\omega^2(x^2+y^2)$$

(i) Write down the expression for the Hamiltonian. [2]

(ii) Show that the Schrodinger equation permits factorization of the wave function and each of the two factors then satisfy a one dimensional oscillator equation. [6]

(iii)Write down the expressions for the eigen functions for the first two states. [2]

Q.5. (a) The Hamiltonian of a system is given by the expression [12]

$$H=\frac{1}{2I_1}(L_x^2+L_y^2)+\frac{1}{2I_3}L_z^2$$

Find an expression for the eigenvalue of the Hamiltonian.
Here L is orbital angular momentum.

(b) An electron in the Coulomb field of a proton with Hamiltonian H is in a state described the wave function

$$\phi=\frac{1}{6}\left[4\psi_{100}+3\psi_{211}-\psi_{210}+\sqrt{10}\,\psi_{21-1}\right]$$

(i) What is the expectation value of the energy? [5]

(ii) What is the expectation value of L^2? [4]

(iii) What is the expectation value of L_z? [4]

Note that $\psi_{nlm}=R_{nl}(r)Y_l^m(\vartheta,\varphi)$ is the eigen function of H with energy $\frac{E_0}{n^2}$

where E_0 is a constant and

n = principal quantum number,

l = angular momentum quantum number

m = projection of angular momentum onto the z-axis..

and $\int\psi^*_{n'l'm'}\psi_{nlm}\,d\tau=\delta_{n'n}\,\delta_{l'l}\,\delta_{m'm}$.

@@@@END OF EXAMINATION@@@@

APPENDIX:

Useful Information:

$\hbar = 1.0546 \times 10^{-34}$ Js , c = velocity of light =2.99792 x 10^8 m s^{-1}
mass of alpha particle = 6.6447 x 10^{-27} kg , 1 eV = 1.6022 x 10^{-19} J

$[A, CD] = [A, C] D + C [A, D]$
$[AC, D] = A [C, D] + [A, D] C$

$[r_i, p_j] = i\hbar\, \delta_{ij}$ where $r_i = (x, y, z)$ and $p_i = (p_x, p_y, p_z)$,
$[L_x, L_y] = i\hbar\, L_z$, $[L_y, L_z] = i\hbar\, L_x$, $[L_z, L_x] = i\hbar\, L_y$ where $\vec{L} = \vec{r} \times \vec{p}$,

The functions $Y_l^m(\vartheta, \varphi)$ are eigen functions of L^2 and L_z operators with the property

$$L^2\, Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2\, Y_\ell^m(\vartheta,\varphi)$$
$$L_z\, Y_\ell^m(\vartheta,\varphi) = m\hbar\, Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_0^\infty exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1}\, exp(-at^2)\, dt = \frac{n!}{2a^{n+1}}$$ with Re a > 0, n =0,1,2,..

$$\int_0^\infty t^{2n}\, exp(-at^2)\, dt = \frac{1.3.5......(2n-1)}{2^{n+1}\, a^n}\sqrt{\frac{\pi}{a}}$$
with Re a > 0 , n=0,1,2.....

$$\int sin^2(x)dx = \frac{x}{2} - \frac{1}{4}sin(2x)$$

$$\int sin(mx)sin(nx)dx = \frac{1}{2}\left[\frac{sin\{(m-n)x\}}{(m-n)} - \frac{sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int sin(mx)cos(nx)dx = -\frac{1}{2}\left[\frac{cos[(m-n)x]}{(m-n)} + \frac{cos[(m+n)x]}{(m+n)}\right]$$

First few Hermite polynomials:

$$H_0(\xi) = 1$$
$$H_1(\xi) = 2\xi$$
$$H_2(\xi) = -2 + 4\xi^2$$

The Hermite polynomials are orthogonal and have the property,

$$\int_{-\infty}^{\infty} H_m(\xi) H_n(\xi)\, e^{-\xi^2}\, d\xi = 0 \quad \textit{for } m \neq n$$

$$\int_{-\infty}^{\infty} [H_n(\xi)]^2\, e^{-\xi^2}\, d\xi = \sqrt{\pi}\; 2^n\, n!$$