UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2006

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS SEVEN PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) From $\begin{cases} \vec{e}_\rho = \vec{e}_x \cos(\phi) + \vec{e}_y \sin(\phi) \\ \vec{e}_\phi = -\vec{e}_x \sin(\phi) + \vec{e}_y \cos(\phi) \end{cases}$ and $\begin{cases} \vec{e}_r = \vec{e}_\rho \sin(\theta) + \vec{e}_z \cos(\theta) \\ \vec{e}_\theta = \vec{e}_\rho \cos(\theta) - \vec{e}_z \sin(\theta) \end{cases}$

deduce that

$$\begin{cases} \vec{e}_x = \vec{e}_\rho \cos(\phi) - \vec{e}_\phi \sin(\phi) \\ \vec{e}_y = \vec{e}_\rho \sin(\phi) + \vec{e}_\phi \cos(\phi) \end{cases} \quad \text{and}$$

$$\begin{cases} \vec{e}_x = \vec{e}_r \sin(\theta)\cos(\phi) + \vec{e}_\theta \cos(\theta)\cos(\phi) - \vec{e}_\phi \sin(\phi) \\ \vec{e}_y = \vec{e}_r \sin(\theta)\sin(\phi) + \vec{e}_\theta \cos(\theta)\sin(\phi) + \vec{e}_\phi \cos(\phi) \\ \vec{e}_z = \vec{e}_r \cos(\theta) \qquad - \vec{e}_\theta \sin(\theta) \end{cases}$$

(10 marks)

(b) (i) Given P(8 , 300^0 , - 6) in a cylindrical coordinate system, find its Cartesian and spherical coordinates . **(6 marks)**

(ii) If a vector field $\vec{F}$ at the point P(8 , 300^0 , - 6) has the value of

$\vec{F} = \vec{e}_\rho(-9) + \vec{e}_\phi 5 + \vec{e}_z(-3)$ in terms of its cylindrical components, then find its corresponding components in Cartesian and spherical coordinates at the same point. (Hint : use the result in (a) and the values of θ *and* ϕ found in (b) (i) to find the answers for (b) (ii)) . **(9 marks)**

Question two

(a) Given a scalar function $f = y^2 - 4x$,

(i) find the magnitude and direction of $\vec{\nabla} f$ at $x = -0.2$ *and* $y = 0$,

(4 marks)

(ii) plot both $f = 0$ *and* $f = 1$ surfaces on $x - y$ *plane* and on the diagram draw the direction and estimate the magnitude of $\vec{\nabla} f$ at $x = -0.2$ *and* $y = 0$. **(4 marks)**

(b) Given $\vec{F} = \vec{e}_\rho 7\rho^2 \cos(\phi) + \vec{e}_\phi 9\rho^2 + \vec{e}_z 5\rho(z+2)$

(i) evaluate the value of $\oint_l \vec{F} \bullet d\vec{l}$ where l : a circle of radius 3 centred at the origin on $z = 0$ (i.e., $x - y$ *plane*) in counter clockwise sense ,

(7 marks)

(ii) Find $\vec{\nabla} \times \vec{F}$ and then evaluate the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by the given closed loop l in (i) . Compare the value here with that obtained in (i) and make a brief remark . **(10 marks)**

Question three

(a) Given the following $2-D$ Laplace equation in spherical coordinate as :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(r,\theta)}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f(r,\theta)}{\partial\theta}\right)=0$$

utilize the separation of variable scheme to split it into two ordinary differential equations. **(8 marks)**

(b) Given the following differential equation as :

$$(1-x^2)\frac{d^2y(x)}{dx^2}-2x\frac{dy(x)}{dx}+12\,y(x)=0$$

utilize the power series method , i.e., setting $y(x)=\sum_{n=0}^{\infty}a_n\,x^{n+s}$ and $a_0\neq 0$,

to find its two independent solutions (up to the a_5 term and set $a_0=1$)

(17 marks)

Question four

The following non-homogeneous differential equation represents a simple harmonic oscillator of mass $m = 2\ kg$ and spring force constant $K = 9\ \frac{N}{m}$ forced to oscillate in a viscous fluid

$$2\frac{d^2 x(t)}{d t^2} - 6\frac{d\, x(t)}{d\, t} + 9\, x(t) = f(t)$$

where $x(t)$: displacement from its resting position

$6\frac{d\, x(t)}{d\, t}$: retardation force by the viscous fluid

$f(t)$: externally applied driving force

(a) Find and write down the general solution to the homogeneous part of the above given differential equation , i.e., $2\ \frac{d^2 x(t)}{d t^2} - 6\ \frac{d\, x(t)}{d\, t} + 9\ x(t) = 0$ **(5 marks)**

(b) If the driving force is given as $f(t) = 9\cos\left(6\, t\right)$, set the particular solution of the given non-homogeneous differential equation as $x(t) = k_1\ \cos\left(6\, t\right) + k_2\ \sin\left(6\, t\right)$ and find the values of k_1 *and* k_2 , **(9 marks)**

(c) (i) Combine the obtained solutions in (a) and (b) to write down the general solution of the given non-homogeneous differential equation , **(2 marks)**

(ii) If the given initial conditions for the system are $x(0) = 5$ and $\left.\frac{d\, x(t)}{d\, t}\right|_{t=0} = 2$ find the values of the arbitrary constants in (c)(i) and thus the specific solution for the given system. **(9 marks)**

Question five

Two simple harmonic oscillators (one is represented by m_1 *and* k_1 and the other represented by m_2 *and* k_2) are jointed together by a spring of spring constant K . The equations of motion for the system are :

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -(k_1 + K)x_1(t) + K x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = K x_1(t) - (k_2 + K)x_2(t) \end{cases}$$

where $x_1(t)$ and $x_2(t)$ are the displacement from their respective resting position .

If $m_1 = 1\ kg$, $m_2 = 2\ kg$, $k_1 = 5\ \dfrac{N}{m}$, $k_2 = 10\ \dfrac{N}{m}$ and $K = 8\ \dfrac{N}{m}$,

(a) show that the coupled differential equations for the system can be simplified to be

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -13\, x_1(t) + 8\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 4\, x_1(t) - 9\, x_2(t) \end{cases}$$

(3 marks)

(b) set $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$, and deduce the following matrix equation :

$$-\omega^2 \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -13 & 8 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

(4 marks)

(c) find the eigenfrequencies ω of the given coupled system , **(6 marks)**

(d) find the eigenvectors of the given coupled system corresponding to each eigenfrequencies found in (c), **(6 marks)**

(e) find the normal coordinates of the given coupled system corresponding to each eigenfrequencies found in (c) . **(6 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin\theta)$	for spherical coordinate system