UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2006

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS
		EACH QUESTION CARRIES 25 MARKS
		MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS 10 PAGES, INCLUDING THIS PAGE.

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Question one

What is meant by a central force? [2 marks]

Define the moment **G**, about the origin, of a non-central force **F** acting on a particle positioned at **r**. Consider the angular momentum, **J**, of the particle at **r** moving with velocity **v**, and show that

$\dot{\mathbf{J}} = \mathbf{G}$. [5 marks]

It follows that for a central force the angular momentum is constant in time. Why is this so? [3 marks]

One of the simplest forms of a central force is $\mathbf{F} = -k\mathbf{r}$, where k is a constant. Show that the particle experiencing such a force orbits the origin in an ellipse. [10 marks]

The Sun has an orbital velocity of about 250 km/s around the centre of the galaxy, whose distance from the Sun is 30,000 light years. Make a rough estimate of the total mass of the galaxy in solar masses, and state any assumptions you make. [5 marks]

mass of the Sun is 2×10^{30} kg

$\sin 2\theta = 2\sin\theta.\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

Question two

State Galileo's principle of relativity. Which assumptions made in the principle are now known to be incorrect? [8 marks]

In classical mechanics Galileo's principle is assumed to be true. As a consequence, the Lagrangian of a closed system of particles cannot depend on time or the spatial co-ordinates that describe the system. What is meant by a closed system? [2 marks]

By consideration of the invariance of the Lagrangian undergoing a Galilean transformation, deduce three conservation laws involving the energy, linear momentum and angular momentum of a closed system of particles.

[15 marks]

The Lagrangian is $L = T - U$ and $\sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] = 0$, where q_i are generalized co-ordinates.

A.(B x C) = C.(A x B) = B.(C x A)

A.(B x C) = -A.(C x B)

A.(B x C) = -B.(A x C) etc.

Question three

Consider a solid body rotating with constant angular velocity ω about a fixed axis. Let vectors **i**, **j** and **k** denote orthogonal unit vectors fixed in the rotating body. Show that if **a** is a vector fixed in the rotating body i.e. that

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

then an observer in an inertial frame finds that

$$\frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} + \omega \times \mathbf{a}$$

where $\dot{\mathbf{a}}$ is the rate of change of **a** as measured by an observer rotating with the solid body. Note that the two observers agree about the rate of change of a scalar quantity. [5 marks]

Obtain the following expression for the absolute acceleration

$$\frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} + 2\omega \times \dot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r}).$$ [5 marks]

The third term on the right hand side is known as the centripetal acceleration. Show that this is directed radially inwards. [3 marks]

The middle term on the right hand side gives rise to the Coriolis force.

Consider the motion of a freely falling body dropped from a small height h above the surface of the Earth. If the Earth were not rotating then its motion would be given by

$$x = 0 \qquad y = 0 \qquad z = h - \tfrac{1}{2}gt^2$$

where **i** is East, **j** is North and the direction of **k** is opposite to g. Because the Earth is rotating the particle will hit the ground ($z = 0$) at a point East of that vertically below its point of release. Derive an expression for this distance in terms of its latitude, the angular velocity of the Earth, h and g. [12 marks]

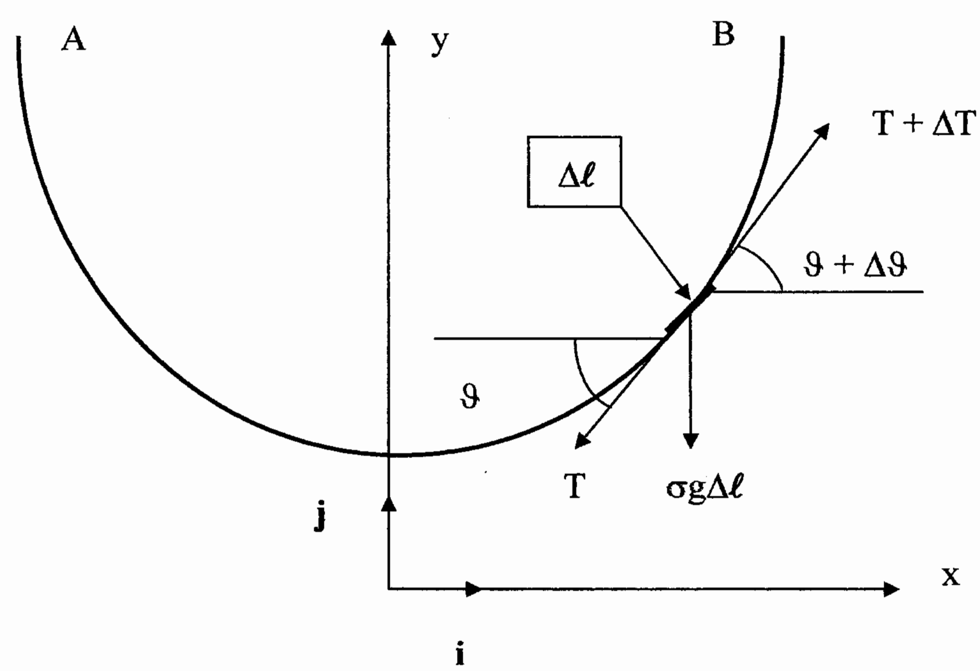
$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = (\boldsymbol{\omega} . \mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}$$

Question four

What is the condition for a particle to be in equilibrium? [2 marks]

If a particle is in equilibrium, and we consider only central forces, what can you deduce about the gradient of the potential energy at the position of the particle? [3 marks]

A uniform chain is suspended from two fixed points A and B at the same horizontal level (see diagram). The linear density of the chain is σ kg m^{-1}, and **i** and **j** are unit vectors in the x and y-directions respectively. By considering the forces acting on a small length of the chain, $\Delta\ell$, derive the equation for the curve in which it hangs. [The curve is known as a *catenary* - from Latin, meaning chain]. [20 marks]



$\sec^2\vartheta = 1 + \tan^2\vartheta$

$\int \sec\vartheta \, d\vartheta = \ln[\sec\vartheta + \tan\vartheta]$

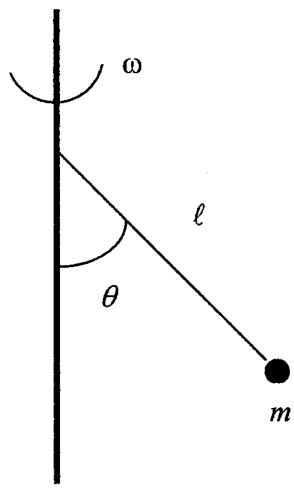
$\int \sec\vartheta \, \tan\vartheta \, d\vartheta = \sec\vartheta$

$\cosh(px) = \frac{1}{2}[\exp(px) + \exp(-px)]$

[hint: resolve forces in the horizontal and vertical directions]

Question five

A pendulum consists of a light rigid rod of length ℓ with a mass m attached at one end. The other end is fixed to a vertical axis in such a way that it can swing freely in the vertical plane. A torque, G, is applied to the axis so that it rotates with constant angular velocity ω about the vertical direction – see the diagram.



The system has two degrees of freedom, the polar co-ordinates θ and φ. For convenience, the direction of the polar axis has been taken to point vertically downwards, so that the position of equilibrium is $\theta = 0$.

Show that the Lagrangian function, $L = T - U$, is

$$L = \frac{1}{2} m\ell^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right) - mg\ell(1 - \cos\theta),$$

where g is the acceleration due to gravity. [5 marks]

Determine Lagrange's equations and hence obtain an expression for $\ddot{\theta}$.

[5 marks]

The magnitude of G is altered until the bob rotates with constant angular velocity, ω, about the vertical axis i.e. $\dot{\varphi} = \omega$. The system now has only one degree of freedom, namely θ. In a frame of reference rotating with the bob the kinetic and potential energies, T' and U' are given by the following expressions.

$$T' = \frac{1}{2} m\ell^2 \dot{\theta}^2$$

$$U' = mg\ell(1 - \cos\theta) - \frac{1}{2} m\ell^2 \omega^2 \sin^2\theta .$$

Explain the changes in form for the expressions for the kinetic and potential energies given earlier in this question. In particular what is the origin of the extra term in U'? [5 marks]

Show that if $\omega^2 > g\ell$, U' has a maximum at both $\theta = 0$ and $\theta = \pi$, and a minimum when $\cos\theta = g/\ell\omega^2$. [5 marks]

Describe the motion of the bob if $E' = T' + U' < 0$. [5 marks]

PHYSICAL CONSTANTS AND UNITS

Acceleration due to gravity	g	9.81 m s^{-2}
Gravitational constant	G	6.673 × 10^{-11} N m^2 kg^{-2}
Avogadro constant	N_A	6.022 × 10^{23} mol^{-1}
	(Note: 1 mole = 1 *gram* molecular-weight)	
Ice point	T_{ice}	273.15 K
Gas constant	R	8.314 J K^{-1} mol^{-1}
Boltzmann constant	k, k_B	1.381 × 10^{-23} J K^{-1} = 0.862 × 10^{-4} eV K^{-1}
Stefan constant	σ	5.670 × 10^{-8} W m^{-2} K^{-4}
Rydberg constant	R_∞	1.097 × 10^7 m^{-1}
	$R_\infty hc$	13.606 eV
Planck constant	h	6.626 × 10^{-34} J s = 4.136 × 10^{-15} eV s
$h/2\pi$	$\hbar$	1.055 × 10^{-34} J s = 6.582 × 10^{-16} eV s
Speed of light *in vacuo*	c	2.998 × 10^8 m s^{-1}
	$\hbar c$	197.3 MeV fm
Charge of proton	e	1.602 × 10^{-19} C
Mass of electron	m_e	9.109 × 10^{-31} kg
Rest energy of electron		0.511 MeV
Mass of proton	m_p	1.673 × 10^{-27} kg
Rest energy of proton		938.3 MeV
One atomic mass unit	u	1.66 × 10^{-27} kg
Atomic mass unit energy equivalent		931.5 MeV
Electric constant	ϵ_0	8.854 × 10^{-12} F m^{-1}
Magnetic constant	μ_0	4π × 10^{-7} H m^{-1}
Bohr magneton	μ_B	9.274 × 10^{-24} A m^2 (J T^{-1})
Nuclear magneton	μ_N	5.051 × 10^{-27} A m^2 (J T^{-1})
Fine-structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	7.297 × 10^{-3} = 1/137.0
Compton wavelength of electron	$\lambda_c = h/mc$	2.426 × 10^{-12} m
Bohr radius	a_0	5.2918 × 10^{-11} m
angstrom	Å	10^{-10} m
torr (mm Hg, 0°C)	torr	133.32 Pa (N m^{-2})
barn	b	10^{-28} m^2

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