UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2006

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **FIVE** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q. 1. (a) What is the de Broglie wavelength of a neutron with kinetic energy of 2.0 MeV? What is its velocity? [5]

(b) Using uncertainty relation show that average potential energy $\langle U \rangle$ for a bound nucleon confined in a nucleus within a sphere of roughly the radius $r_0 = 1.2 \times 10^{-15}\ m$ is roughly given by the relation [5]

$$-\langle U \rangle \geq 10 \text{ MeV}$$

(c) Write short note on

(i) The degenerate states. [2]

(ii) Parity. [2]

(iii) Complete set of states. [2]

(d) The wave function of a particle moving in one dimension is given by:

$$\psi(x) = 0 \quad for\ x < 0$$
$$= B\sqrt{x}\exp(-\beta x) \quad for\ x \geq 0$$

where β is a real and positive constant.

(i) Calculate the normalization constant B. (It is a function of β.) [4]

(ii) Calculate the average position of the particle on the *x* axis, as a function of β. [5]

Q.2. (a) Take the wavefunction $\varphi(x) = \frac{1}{\sqrt{2\pi\hbar}} exp(ikx)$ [2]

Calculate the effect of the operator $[x\ ,\ p_x\ ]$ on φ (x).

(b) For each of the following functions of x, decide whether they have even, odd or no definite parity.

(i) $x\,exp(-\alpha x^2)$ where α is a constant. [1]

(ii) $Ax + B$ where A and B are constants. [1]

(iii) $x^2 cos(kx)$ where k is a constant. [1]

(c) Consider a particle confined in a one-dimensional box (i.e. in a potential)

$$V(x)=0 \text{ for } 0 < x < L$$
$$= \infty \text{ for } x \leq 0 \text{ and } x \geq L .$$

(i) Determine the solution $\varphi_n(x)$ and energy E of the stationary Schrodinger equation for this problem [15]

(ii) Normalize the wave function $\varphi_n(x)$. [2]

(iii) Write down the time dependent wave function $\psi_n(x,t)$ for the n^{th} stationary state in this potential. [3]

Q.3. (a) Show that $[x^2, p_x^2] = 4i\hbar x p_x + 2\hbar^2$ [4]

(b) Orbital angular momentum operator is defined by $\vec{L} = \vec{r} \times \vec{p}$.
Show that
(i) $[L_z, x] = i\hbar y$ [4]
(ii) $[L_z, y] = -i\hbar x$ [2]
(iii) $[L_z, z] = 0$ [2]
(iv) Using the above identities show that $[L_z, r^2] = 0$ [4]
where $r^2 = x^2 + y^2 + z^2$.

(c) A Hamiltonian H is defined in terms of operators A and $A^\dagger$ by

$$H = \omega A^\dagger A + \tfrac{1}{2}\hbar\omega$$

and $Hu_E = E\,u_E$ where E is energy of the system defined by H.
Given the property $[A, A^\dagger] = \hbar$ and a function $v_E = A\,u_E$,
show that

(i) $[H, A] = -\hbar\omega A$ [4]
(ii) v_E belongs to energy $E - \hbar\omega$. [5]

Q.4. (a) For central potentials V(r), the radial part of the Schrodinger equation for orbital angular momentum $\ell = 0$ is given by

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2m}{\hbar^2}[E - V(r)]R = 0$$

where m is the reduced mass.
With S(r)=r R(r), the equation reduces to

$$\frac{d^2S}{dr^2} + \frac{2m}{\hbar^2}[E - V(r)]S = 0$$

The potential between proton and neutron, which binds the two particles is approximated by the central potential

$$V(r) = -V_0 \qquad r < a$$
$$= 0 \qquad r \geq a$$

where a is the range of the potential.

(i) State the boundary conditions on S(r) for bound states. [2]
(ii) What are the continuity conditions at r = a. [2]
(iii) Find the acceptable solutions for $r < a$ and $r \geq a$. [10]
(iv) Explain how E can be calculated using the acceptable solutions for a given value of a and V_0. [5]

(b) Given the functions

$$\phi_1(x) = a\,e^{-x} + b\,e^{-2x} \quad and \quad \phi_2(x) = a\,e^{-x} - b\,e^{-2x}$$

where a and b are real constants and $0 < x < \infty$.
(i) Show that $\phi_1(x)$ and $\phi_2(x)$ are orthogonal if

$$\frac{1}{2}a^2 - \frac{1}{4}b^2 = 0.$$ [3]

(ii) Use the condition of (i) to find the normalization constant for the function $\phi_1(x)$. [3]

Q.5. (a) The potential energy of a 3-dimensional harmonic oscillator is given by

$$V = \frac{1}{2}m[\omega_1^2 x^2 + \omega_2^2 (y^2 + z^2)] \qquad \text{with } \omega_1 > \omega_2$$

The energy eigen-values are given by the relation

$$E = (n_1 + \tfrac{1}{2})\hbar\omega_1 + (n_2 + n_3 + 1)\hbar\omega_2$$

where n_1 , n_2 , n_3 = 0, 1,2 ,. . .

(i)	Determine E for its ground state.	[1]
(ii)	Determine E for its first excited state.	[2]
(iii)	How many states belong to 1st excited state.	[2]
(iv)	Are they degenerate?	[1]

(b) The solutions of Schrodinger equation for a hydrogen like atom are given by the wave function

$$\psi_{n\ell m} = R_{n\ell}(r) Y_\ell^m(\vartheta, \varphi)$$

where $R_{n\ell}(r) =$ radial functions.

$Y_\ell^m(\vartheta, \varphi) =$ spherical harmonic functions.

The energy eigen values are given by

$$E_n = -\frac{\beta}{n^2}$$

where β = constant , n = 1 , 2 , 3 , ..., ℓ = 0 , 1 , 2 , . . n-1.
and $-\ell < m < \ell$.

(i) How many states belong to the 2nd excited state? [4]
Specify n , ℓ , and m values for each of the states.

(ii) A state of the hydrogen like atom under the influence of external force is described by the eigen function

$$\varphi = A\,[3\psi_{100} - \psi_{210} + \sqrt{2}\,\psi_{211} - \sqrt{5}\,\psi_{21-1}]$$

(a) Find A by normalizing the eigen function φ . [5]
(b) What is the expectation value of L^2 ? [5]
(c) What is the expectation value of L_z ? [5]

Note: $\int \psi^*_{n_1\ell_1 m_1} \psi_{n_2\ell_2 m_2}\, d\tau = \delta_{n_1 n_2}\, \delta_{\ell_1\ell_2}\, \delta_{m_1 m_2}$

@@@@END OF EXAMINATION@@@@

APPENDIX:

Given: $\hbar = 1.0546 \times 10^{-34}$ Js , c = *velocity of light* =2.99792 x 10^{8} $m\,s^{-1}$

mass of neutron/proton = 1.6749 x 10^{-27} *kg* , k=1.3807x 10^{-23} JK^{-1} .

1ev=1.6022 x 10^{-19} J.

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

$[\, r_i , p_j \,] = i\hbar\, \delta_{ij}$ where $r_i = (x, y, z)$ and $p_i = (p_x, p_y, p_z)$,

$[\, L_x , L_y \,] = i\hbar\, L_z$, $[\, L_y , L_z \,] = i\hbar\, L_x$, $[\, L_z , L_x \,] = i\hbar\, L_y$ where $\vec{L} = \vec{r} \times \vec{p}$,

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2 Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\vartheta,\varphi)$$

$$L_z Y_\ell^m(\vartheta,\varphi) = m\hbar Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_0^\infty exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1} exp(-at^2)\,dt = \frac{n!}{2a^{n+1}} \qquad \text{with Re } a > 0, \; n = 0,1,2,..$$

$$\int_0^\infty t^{2n} exp(-at^2)\,dt = \frac{1.3.5......(2n-1)}{2^{n+1} a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....

$$\int sin^2(x)dx = \frac{x}{2} - \frac{1}{4} sin(2x)$$

$$\int sin(mx) sin(nx)\,dx = \frac{1}{2}\left[\frac{sin\{(m-n)x\}}{(m-n)} - \frac{sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int sin(mx) cos(nx)dx = -\frac{1}{2}\left[\frac{cos[(m-n)x]}{(m-n)} + \frac{cos[(m+n)x]}{(m+n)}\right]$$

$$\int_{-\infty}^{\infty} H_n(\xi) H_m(\xi) exp(-\xi^2)d\xi = \pi^{\frac{1}{2}}\, 2^n\, n!\, \delta_{nm}$$ where H(ξ) are Hermite polynomials and are real.

$$\int_0^\infty t^{z-1} \exp(-kt)dt = k^{-z}\,\Gamma(z) \qquad \text{Re}\, z > 0, \text{Re}\, k > 0.$$

$$\Gamma(n+1) = n! \quad \textit{for } n = 1,2,\ldots \textit{ and } \Gamma(1) = 1.$$

$$\int x^n e^{-m\alpha x}\, dx = \frac{\Gamma(n+1)}{(m\alpha)^{n+1}} \qquad \textit{for } m > 0 \textit{ and } n \geq 0.$$

You can calculate the integrals you need by expressing powers of x through (repeated) differentiation with respect to the parameter in the exponential, e.g.

$$\int_a^b dx\, x \exp(-\gamma x) = -\frac{\partial}{\partial\gamma}\int_a^b dx \exp(-\gamma x) \textit{ and } \int_a^b dx\, x^2 \exp(-\gamma x) = \frac{\partial^2}{\partial\gamma^2}\int_a^b dx \exp(-\gamma x) \textit{ and so on.)}.$$