UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2006/2007

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) Given P(9 , 130^0 , 290^0) in a spherical coordinate system, find its Cartesian and cylindrical coordinates . **(6 marks)**

(b) Given the following differential equation as :

$$4\,x\,\frac{d^2\,y(x)}{d\,x^2} + 2\,\frac{d\,y(x)}{d\,x} + y(x) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ and $a_0 \neq 0$,

to find its two independent solutions (up to the a_5 term and set $a_0 = 1$)

(19 marks)

Question two

(a) Given $\vec{F} = \vec{e}_x\, 3y + \vec{e}_y\, 3x + \vec{e}_z\, 2z$,

(i) find $\vec{\nabla} \times \vec{F}$, **(3 marks)**

(ii) evaluate the value of the line integral $\int_L \vec{F} \bullet d\vec{l}$ if L: the straight line segment from $(0,0)$ to $(6,8)$ on $z = 0$ plane. If L is any other line path from $(0,0,0)$ to $(6,8,0)$, would it yields different values for the line integral $\int_L \vec{F} \bullet d\vec{l}$? Why ? **(9 marks)**

(b) Given $\vec{F} = \vec{e}_\rho\, 3\rho^3 + \vec{e}_\phi\, 5\rho^2 z + \vec{e}_z\, 6\rho\, z^2$,

(i) find $\vec{\nabla} \bullet \vec{F}$, **(4 marks)**

(ii) evaluate the value of the closed surface integral $\oiint_S \vec{F} \bullet d\vec{s}$ if S : the surface of a cylinder with a height of $h = 3$, a cross-section radius of $\rho_0 = 2$ and its centre line coincides with the $z-axis$, i.e., $S = S_1 + S_2 + S_3$

where $S_1 : z = 0, d\vec{s} = -\vec{e}_z\, \rho\, d\rho\, d\phi, 0 \le \rho \le 2, 0 \le \phi \le 2\pi$,

$S_2 : z = 3, d\vec{s} = +\vec{e}_z\, \rho\, d\rho\, d\phi, 0 \le \rho \le 2, 0 \le \phi \le 2\pi$,

$S_3 : \rho = 2 \;,\; d\vec{s} = +\vec{e}_\rho\, 2\, dz\, d\phi \;,\; 0 \le z \le 3 \;,\; 0 \le \phi \le 2\pi$.

(9 marks)

(Hint : can utilize the divergence theorem to find the value of $\oiint_S \vec{F} \bullet d\vec{s}$)

Question three

An elastic string of length L is fixed at its two ends, i.e., at $x = 0$ and $x = L$. Given one-dimensional wave equation governing the transverse deflection of the string $u(x,t)$ as :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$ where c is a constant relating to wave velocity

(a) Set $u(x,t) = F(x)\,G(t)$ and use the separation of variables scheme to deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = -k^2 F(x) & \cdots\cdots \quad (1) \\ \dfrac{d^2 G(t)}{d t^2} = -k^2 c^2 G(t) & \cdots\cdots \quad (2) \end{cases}$$

where k^2 is a separation constant. (6 marks)

(b) Given the general solution to eq.(1) and eq.(2) in (a) as :

$$u(x,t) = \sum_{for\ all\ k} u_k(x,t)$$ where

$$u_k(x,t)$$
$$= (A_k \cos(k x) + B_k \cos(k x))\,(C_k \cos(c k t) + D_k \cos(c k t))$$

where A_k , B_k , C_k *and* D_k are arbitrary constants.

Applying two fixed end conditions and zero initial velocity condition, i.e.,

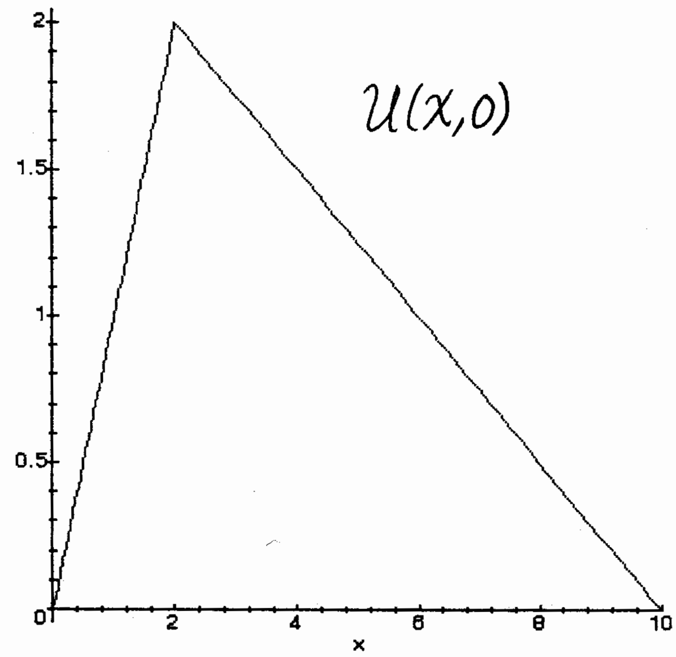
$u_k(0,t) = 0$, $u_k(L,t) = 0$ and $\left.\dfrac{\partial u_k(x,t)}{\partial t}\right|_{t=0} = 0$, deduce that

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{c n \pi t}{L}\right)$$ where E_n is an arbitrary constant

(9 marks)

Question three (continued)

(c) If $c = 2$, $L = 10$ and the initial values of $u(x, t)$ is given as :



find E_n in terms of n and calculate the values of E_1 , E_2 *and* E_3 .(10 marks)

(hint : $\int_{x=0}^{L} \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L})\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases}$)

Question four

The following non-homogeneous differential equation represents a simple harmonic oscillator of mass $m = 2\ kg$ and spring force constant $K = 5\ \frac{N}{m}$ forced to oscillate in a viscous fluid

$$2\frac{d^2 x(t)}{d t^2} - 2\frac{d\, x(t)}{d\, t} + 5\, x(t) = f(t)$$

where $x(t)$: displacement from its resting position

$2\frac{d\, x(t)}{d\, t}$: retardation force by the viscous fluid

$f(t)$: externally applied driving force

(a) Find and write down the general solution to the homogeneous part of the above given differential equation, i.e., $2\frac{d^2 x(t)}{d t^2} - 2\frac{d\, x(t)}{d\, t} + 5\, x(t) = 0$ **(5 marks)**

(b) If the driving force is given as $f(t) = 20\,t - 25\,t^2$, set the particular solution of the given non-homogeneous differential equation as $x(t) = k_1 + k_2\, t + k_3\, t^2$ and find the values of k_1 , k_2 *and* k_3 , **(9 marks)**

(c) (i) Combine the obtained solutions in (a) and (b) to write down the general solution of the given non-homogeneous differential equation , **(2 marks)**

(ii) If the given initial conditions for the system are $x(0) = 9$ and $\left.\frac{d\, x(t)}{d\, t}\right|_{t=0} = 4$ find the values of the arbitrary constants in (c)(i) and thus the specific solution for the given system. **(9 marks)**

Question five

Two simple harmonic oscillators (one is represented by m_1 *and* k_1 and the other represented by m_2 *and* k_2) are joined together by a spring of spring constant K . The equations of motion for the system are :

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -(k_1 + K)\, x_1(t) + K\, x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = K\, x_1(t) - (k_2 + K)\, x_2(t) \end{cases}$$

where $x_1(t)$ and $x_2(t)$ are the displacement from their respective resting position .

If $m_1 = 1\ kg$, $m_2 = 3\ kg$, $k_1 = 2\ \dfrac{N}{m}$, $k_2 = 6\ \dfrac{N}{m}$ and $K = 9\ \dfrac{N}{m}$,

(a) set $x_1(t) = X_1\, e^{i\omega t}$ and $x_2(t) = X_2\, e^{i\omega t}$, and deduce the following matrix equation :

$$-\omega^2 \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -11 & 9 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

(5 marks)

(b) find the eigenfrequencies ω of the given coupled system , **(7 marks)**

(c) find the eigenvectors of the given coupled system corresponding to each eigenfrequencies found in (b), **(7 marks)**

(d) find the normal coordinates of the given coupled system corresponding to each eigenfrequencies found in (b) . **(6 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right)$$

$$+ \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3) represents (x, y, z) for rectangular coordinate system

represents (ρ, ϕ, z) for cylindrical coordinate system

represents (r, θ, ϕ) for spherical coordinate system

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ represents $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ for rectangular coordinate system

represents $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$ for cylindrical coordinate system

represents $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ for spherical coordinate system

(h_1, h_2, h_3) represents $(1, 1, 1)$ for rectangular coordinate system

represents $(1, \rho, 1)$ for cylindrical coordinate system

represents $(1, r, r\sin\theta)$ for spherical coordinate system