UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2006/2007

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS EIGHT PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

P272 MATHEMATICAL METHODS FOR PHYSICISTS

Question one

(a) (i) Given the rectangular coordinates of a point P as $(-3\ ,\ -4\ ,\ 5)$, find its cylindrical and spherical coordinates respectively. Express the answers of angles in degrees. **(4 marks)**

(ii) Given the spherical coordinates of a point P as $(7\ ,120^0\ ,225^0)$, find its cylindrical and rectangular coordinates respectively. **(4 marks)**

(b) For a point P on $x-y\ plane$, i.e., $z=0$,

(i) draw the rectangular unit vectors $\vec{e}_x$, $\vec{e}_y$ as well as the cylindrical unit vectors $\vec{e}_\rho$, $\vec{e}_\phi$ for the given point on $x-y\ plane$, **(3 marks)**

(ii) express $\vec{e}_\rho$, $\vec{e}_\phi$ in terms of $\vec{e}_x$, $\vec{e}_y$ and deduce that

$d\,\vec{e}_\phi = -\,\vec{e}_\rho\, d\,\phi$ and $d\,\vec{e}_\rho = \vec{e}_\phi\, d\,\phi$ **(5 marks)**

(c) Given $f(r,\theta,\phi) = r^2\,(2-\cos(\theta)\sin(\phi))$,

(i) find $\vec{\nabla} f$, **(3 marks)**

(ii) evaluate $\vec{\nabla} f$ at the point $P:(4\,,120^0\,,270^0)$ and also find the directional derivative of f along the direction of $\vec{e}_r\,4+\vec{e}_\theta\,3$. **(6 marks)**

Question two

(a) Given any vector function $\vec{F}$ in Cartesian coordinate system , (i.e.,

$\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z$ where F_x , F_y *and* F_z are all functions of

(x, y, z)),verify the following identity :

$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$ **(6 marks)**

(b) Given a vector field $\vec{G}(\rho, \phi, z) = \vec{e}_\rho \rho^3 + \vec{e}_\phi \rho^2 z \cos(\phi) + \vec{e}_z \rho z^2$,

(i) carry out the following closed surface integration of $\oiint_S \vec{G} \bullet d\vec{s}$

where S : the surface enclose the whole of a cylindrical tube of radius ρ_0

and height h , with $z - axis$ coincides with the axial line of the tube , i.e.,

$S = S_1 + S_2 + S_3$ where

S_1 : circular disk surface of radius ρ_0 on $z = 0$ plane

S_2 : circular disk surface of radius ρ_0 on $z = h$ plane

S_3 : circular tube surface of radius ρ_0 on $\rho = \rho_0$ plane with height h

Express your answer in terms of ρ_0 and h . **(12 marks)**

(ii) carry out the value integral of $\iiint_V (\vec{\nabla} \bullet \vec{G})\, dv$ where V : the volume of the given cylindrical tube , i.e., the volume enclosed by the closed surface S specified in (b)(i) . Compare it with that obtained in (b)(i) and make brief comments. **(7 marks)**

Question three

If the transverse wave amplitude function $u(x,t)$ of a certain vibrating string follows the following partial differential equation : $\dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{1}{9}\dfrac{\partial^2 u(x,t)}{\partial t^2} = 0$,

(a) set $u(x,t) = X(x)\,T(t)$ and utilize the separation variable scheme to deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 X(x)}{dx^2} = -\,k^2\,X(x) \\ \dfrac{d^2 T(t)}{dt^2} = -\,9\,k^2\,T(t) \end{cases}$$

where k is a separation constant , (**4 marks**)

(b) (i) by direct substitution, show that $X(x) = A_k \cos(k\,x) + B_k \sin(k\,x)$ and $T(t) = C_k \cos(3\,k\,t) + D_k \sin(3\,k\,t)$ are a general solution to the ordinary differential equations in (a) with A_k , B_k , C_k *and* D_k as arbitrary constants,

(**3 marks**)

(ii) given the length of the vibrating string as six metres with both ends fixed , i.e., $u(0,t) = 0 = u(6,t)$, find the eigenvalues of k and write down the general solution of $u(x,t)$ to include all the eigenvalues of k , (**6 marks**)

(c) given the initial condition as $\left.\dfrac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0$ and

$$u(x,0) = \begin{cases} \dfrac{x}{2} & \text{for } 0 \le x \le 4 \\ -x+6 & \text{for } 4 \le x \le 6 \end{cases}$$

, determine the specific values of those arbitrary constants in the general solution of $u(x,t)$ written down in (b)(ii) and thus write down the specific solution of this given problem.

(hint : $\displaystyle\int_0^6 \sin(\frac{n\pi}{6}x)\,\sin(\frac{m\pi}{6}x)\,dx = \begin{cases} 3 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$

where n *and* m are non-zero positive integers) **(12 marks)**

Question four

Given the following differential equation $\frac{d^2 y(x)}{d x^2} - 3 \frac{d y(x)}{d x} + 2 y(x) = 0$,

using the power series method , i.e., set $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ with $a_0 \neq 0$ and substituting it back to the given differential equation ,

(a) requiring the coefficients of the lowest power terms for x , i.e., x^{s-2} and x^{s-1} , to be zero and thus write down the indicial equations. From these equations find the values of s (possibly also the values of a_1 from setting $a_0 = 1$),

(6 marks)

(b) requiring the coefficients of all the rest power terms for x , i.e., x^{s+n} with $n = 0, 1, 2, 3, \cdots$, to be zero and find the recurrence relation, **(5 marks)**

(c) (i) using the recurrence relation in (b) , find the values of a_2 , a_3 ,$\cdots$ a_6 if $a_0 = 1$ for each value of s found in (a) . **(12 marks)**

(ii) write down the general solution of the given differential equation. **(2 marks)**

Question five

(a) Given $m\dfrac{d^2 x}{dt^2} = -kx$, and $m = \dfrac{1}{3}\ kg$ & $k = 12\ \dfrac{N}{m}$

(i) find the values of the angular frequency, frequency and period of the given simple harmonic oscillator system, **(3 marks)**

(ii) write down the general solution of the given problem. **(2 marks)**

(b) Two simple harmonic oscillators (one is represented by m_1 and k_1 and the other represented by m_2 and k_2) are jointed together by a spring of spring constant k_3.

The coupled differential equations are simplified to be :

$$\begin{cases} \dfrac{d^2 x_1}{dt^2} = -16x_1 + 12x_2 \\ \dfrac{d^2 x_2}{dt^2} = 3x_1 - 7x_2 \end{cases}$$

(i) set $x_1(t) = X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$, deduce the following matrix equation $\lambda X = AX$

where $\lambda = -\omega^2$, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $A = \begin{pmatrix} -16 & 12 \\ 3 & -7 \end{pmatrix}$

(4 marks)

(ii) find the eigenfrequencies ω for the matrix equation in (b)(i) **(6 marks)**

(iii) find the eigenvectors corresponding to the eigenfrequencies found in (b)(ii) respectively, **(5 marks)**

(iv) find the normal coordinates of the system. **(5 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1 , u_2 , u_3) represents (x , y , z) for rectangular coordinate system

represents (ρ , ϕ , z) for cylindrical coordinate system

represents (r , θ , ϕ) for spherical coordinate system

$(\vec{e}_1 , \vec{e}_2 , \vec{e}_3)$ represents $(\vec{e}_x , \vec{e}_y , \vec{e}_z)$ for rectangular coordinate system

represents $(\vec{e}_\rho , \vec{e}_\phi , \vec{e}_z)$ for cylindrical coordinate system

represents $(\vec{e}_r , \vec{e}_\theta , \vec{e}_\phi)$ for spherical coordinate system

(h_1 , h_2 , h_3) represents $(1 , 1 , 1)$ for rectangular coordinate system

represents $(1 , \rho , 1)$ for cylindrical coordinate system

represents $(1 , r , r\sin\theta)$ for spherical coordinate system