UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2006/2007

TITLE OF THE PAPER: QUANTUM MECHANICS

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **FIVE** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1.

(a) A typical thermal neutron kinetic energy equals $\frac{3}{2}kT$ at T=300K.

What is its velocity and its de-Broglie wavelength? [5]

(b) Using the uncertainty relation, estimate the radius of the electron whose ionization energy is 13.6 eV. [5]

(c) Explain how following experiments can be understood as quantum phenomena.

(i) Photoelectric effect. [2]

(ii) Black body radiation spectrum. [2]

(iii) Compton scattering. [2]

(d) Take the wave function $\varphi(x) = N\exp(-\mu x^2)$

(i) Calculate the effect of the operator $x\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$ on $\varphi(x)$. [2]

(ii) Calculate the effect of the operator $\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)x$ on $\varphi(x)$. [2]

(iii) Calculate the difference of the two previous results,
i.e. {result of (a) - result of (b)}.
Express the answer in terms of $\varphi(x)$. [2]

(e) Given $\psi = a_1\phi_1 + a_2\phi_2$ and

$$A\phi_1 = \alpha\phi_1$$

$$B\phi_2 = \beta\phi_2$$

where ϕ_1 and ϕ_2 are ortho-normal functions. Here A and B are linear operators.
What is the result of

(i) $(A+B)\psi$ **(ii)** $(A-B)\psi$ **(iii)** $AB\psi$ [3]

Q.2.

A one dimensional harmonic oscillator is in a state such that at time t,

$$\psi(x,t) = A\exp(-x^2/2a^2)\exp(-i\,p_0 t/\hbar) \quad \text{for } -\infty \le x \le \infty$$

where a and p_0 are positive constants.

Find

(i) Probability of finding the particle at any x. [2]

(ii) Normalization constant A. [2]

(iii) The expectation value of x. [3]

(iv) The variance $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$. [5]

(v) Use the Schrodinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$$

to derive an expression for the potential. [5]

(vi) Use the expression of (v) to find the expectation value of the potential. [8]

Q.3. Consider the one dimensional problem of a particle of mass m in a potential

$$V = \infty , \quad x < 0,$$
$$= 0 \quad 0 \le x \le a,$$
$$= V_0 \quad x > a.$$

(i) Sketch the potential, [2]

(ii) Find the solutions in the three regions, [6]

(iii) Show that the bound state energies ($E < V_0$) are given by the equation [13]

$$\tan\left(\frac{\sqrt{2mE}\, a}{\hbar}\right) = -\sqrt{\frac{E}{V_0 - E}}$$

Explain how the ground state and higher state energies are determined for a given potential V_0 .

(iv) Write the normalization condition. [2]
(Do not attempt to evaluate the integral).

(v) Sketch the ground state wave function. [2]

Q.4.

(a) Explain the following:

(i) What is the difference between a state of the system given by the ket $|\ell m\rangle$ and the wave function $\varphi_{\ell m}(r,\vartheta,\varphi)$ describing the same state. [2]

(ii) A dynamical quantity is always represented by a Hermitian linear operator. [2]

(b) Show that the eigen-functions of a Hermitian operator corresponding to two different eigenvalues are orthogonal. [5]

(c) Using the relation $[x_i, p_j] = i\hbar\delta_{ij}$, where $i, j = x,y,z$ [6]
show that $[x^2, p_x^2] = 4i\hbar\, x p_x + 2\hbar^2$

(d) A Hamiltonian H is defined in terms of operators A and $A^\dagger$ by
$H = \omega A^\dagger A + \frac{1}{2}\hbar\omega$
and $Hu_E = E u_E$ where E is energy of the system defined by H.
Given the property $[A, A^\dagger] = \hbar$ and a function $v_E = A^+ u_E$,
show that

(i) $[H, A^+] = \hbar\omega A^+$ [5]

(ii) Use the result of (i) to show that v_E belongs to energy $E + \hbar\omega$. [5]

Q.5.

(a) The Hamiltonian of a rotating system with moment of inertia I is given by the expression

$$H = \frac{1}{2I}(L_x^2 + L_y^2)$$

where $\vec{L} = \vec{r} \times \vec{p}$.

(i) Show that $y_\ell^m(\vartheta,\varphi)$ are eigen-functions of H . [5]

(ii) Determine the eigen-value of H . [5]

(b) An electron is described by an Hamiltonian $H = H_0 + H_1$ where $H_0 = \frac{p^2}{2m} - \frac{e^2}{r}$ and H_1 describes the contribution from external force.

H_0 has the eigen functions $\psi_{nlm} = R_{nl}(r) Y_l^m(\vartheta, \varphi)$ and energy E_n.

Here

n = principal quantum number,
l = angular momentum quantum number
m = projection of angular momentum.

The eigen functions $\psi_{n\ell m}$ have following properties:

$$H_0 \psi_{n\ell m} = E_n \psi_{n\ell m}$$

$$H_1 \psi_{n\ell m} = \alpha \psi_{n\ell m}$$

A state of the electron is described by eigen-function $\phi = N(\psi_{100} - \sqrt{2}\psi_{210})$ and the eigen energy for the electron is given by $H\varphi = E\varphi$.

(i) Determine the normalization constant N. [5]

(ii) Determine the expectation value of H. [10]

Note: $\int \psi^*_{n_1\ell_1 m_1} \psi_{n_2\ell_2 m_2} \, d\tau = \delta_{n_1 n_2} \delta_{\ell_1\ell_2} \delta_{m_1 m_2}$

@@@@END OF EXAMINATION@@@@

APPENDIX:

Given: $\hbar = 1.0546 \times 10^{-34}$ *J s* , *c = velocity of light* $= 2.99792 \times 10^{8}$ *m s*$^{-1}$

mass of neutron/proton = 1.6749×10^{-27} *kg* , $k = 1.3807 \times 10^{-23}$ *JK*$^{-1}$.

1ev = 1.6022×10^{-19} *J.* , *mass of electron* = 9.10938×10^{-31} *kg.*

Useful Information:

$[A, CD] = [A, C] D + C [A, D]$
$[AC, D] = A [C, D] + [A, D] C$

$[r_i, p_j] = i\hbar\, \delta_{ij}$ where $r_i = (x, y, z)$ and $p_i = (p_x, p_y, p_z)$,
$[L_x, L_y] = i\hbar\, L_z$, $[L_y, L_z] = i\hbar\, L_x$, $[L_z, L_x] = i\hbar\, L_y$ where $\vec{L} = \vec{r} \times \vec{p}$,

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2 Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\vartheta,\varphi)$$

$$L_z Y_\ell^m(\vartheta,\varphi) = m\hbar Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_{-\infty}^{\infty} dz\, e^{-\alpha z^2} = \sqrt{\frac{\pi}{\alpha}} \quad , \quad \int_{-\infty}^{\infty} dz\, z^2 e^{-\alpha z^2} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \quad , \quad \int_{-\infty}^{\infty} dz\, z^4 e^{-\alpha z^2} = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$

$$\int_0^\infty \exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1} \exp(-at^2)\, dt = \frac{n!}{2a^{n+1}}$$ with Re a > 0, n =0,1,2,..

$$\int_0^\infty t^{2n} \exp(-at^2)\, dt = \frac{1.3.5......(2n-1)}{2^{n+1} a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....

$$\int \sin^2(x)dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int \sin(mx)\sin(nx)dx = \frac{1}{2}\left[\frac{\sin\{(m-n)x\}}{(m-n)} - \frac{\sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int \sin(mx)\cos(nx)dx = -\frac{1}{2}\left[\frac{\cos[(m-n)x]}{(m-n)} + \frac{\cos[(m+n)x]}{(m+n)}\right]$$

$$\int_{-\infty}^{\infty} H_n(\xi)\, H_m(\xi) \exp(-\xi^2)d\xi = \pi^{\frac{1}{2}}\, 2^n\, n!\, \delta_{nm}$$ where $H(\xi)$ are Hermite polynomials and are real.

$$\int_0^\infty t^{z-1}\exp(-kt)dt = k^{-z}\,\Gamma(z) \quad \mathrm{Re}\, z > 0, \mathrm{Re}\, k > 0.$$

$$\Gamma(n+1) = n! \quad \text{for } n = 1,2,\ldots \text{ and } \Gamma(1) = 1.$$

$$\int x^n e^{-m\alpha x}\, dx = \frac{\Gamma(n+1)}{(m\alpha)^{n+1}} \quad \text{for } m > 0 \text{ and } n \geq 0.$$

You can calculate the integrals you need by expressing powers of x through (repeated) differentiation with respect to the parameter in the exponential, e.g.

$$\int_a^b dx\, x\exp(-\gamma x) = -\frac{\partial}{\partial\gamma}\int_a^b dx \exp(-\gamma x) \text{ and } \int_a^b dx\, x^2\exp(-\gamma x) = \frac{\partial^2}{\partial\gamma^2}\int_a^b dx \exp(-\gamma x) \text{ and so on.).}$$