UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2006/07

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **FIVE** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1.

(A) What is the de-Broglie wavelength of neutron with kinetic energy of 1.7 MeV? What is its velocity? Will such neutrons produce diffraction pattern in a crystal of lattice point distances of the order of 10^{-8} m? [5]

(B) Using the uncertainty relation, show that in a nucleus with the average potential energy $\langle U \rangle \geq 15$ Mev, the bound nucleon is confined within a sphere of radius $r_0 \geq 1.2 \times 10^{-15}$ m. [4]

(C) Explain [8]

(i) Parity.
(ii) Constant of motion in quantum mechanics.
(iii) Probability interpretation of wave function.
(iv) Complete set

(D) Show that [8]

$$\Psi_k(x,t) = (Ae^{ikx} + Be^{-ikx})\,e^{-iE_k t/\hbar}$$

is a solution of the one-dimensional time-dependent Schrödinger equation for a particle of mass m and energy E_k with potential V(x) = 0. Here A, B are constants and $k^2 = \frac{2mE_k}{\hbar^2}$. Show that the probability current corresponding to $\Psi_k(x,t)$ equals

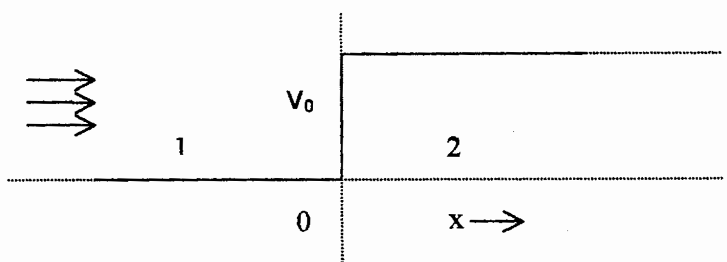
$$j(x,t) = \frac{k\hbar}{m}(|A|^2 - |B|^2).$$

What is the interpretation of this?

Note: For a one dimensional problem probability current $j(x,t) = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$

Q.2. Consider the step potential

$$V(x) = V_0 \qquad x > 0$$
$$= 0 \qquad x < 0$$



Consider a current of particles of mass *m* propagating from left to right of energy $E > V_0$.

Define $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$, $k_2 = \sqrt{\frac{2m(E-V)}{\hbar^2}}$

Then the general solutions for the regions *1 (x<0)* and *2(x >0)* are

$$\phi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \quad , \quad \phi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$$

(i) State the boundary conditions on the solutions. [4]

(ii) Show that $B_2 = 0$ and $A_1 + B_1 = A_2$. [3]

(iii) Show that $\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$ *and* $\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$. [10]

(iv) Show that the probability current density

$$j(x) = \frac{\hbar}{2mi}\left[\phi^*(x)\frac{\partial\phi(x)}{\partial x} - \phi(x)\frac{\partial\phi^*(x)}{\partial x}\right] = \frac{\hbar k_2}{m}|A_2|^2$$ [6]

(v) Do the solutions have any definite parity ? [2]

Q.3. Verify that the two wave functions [8]

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

and

$$\phi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

are solutions of the eigenvalue problem

$$\hat{H}\phi_n(x) = E_n\phi_n(x) \quad \text{with} \quad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$$

(i) Determine E_n for each of them. [5]
(ii) What is the parity of each state. [2]
(iii) Determine the solutions $\phi_0(x,y,z)$ and $\phi_1(x,y,z)$ for the Hamiltonian [10]

$$\hat{H} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] + \frac{m\omega^2}{2}\left[x^2 + y^2 + z^2\right]$$

and E_n for each of them.

Q.4.
(A) show that

(i) $[f(\vec{r}), p_x] = i\hbar\frac{\partial}{\partial x}f(\vec{r})$ where $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$. [3]

(ii) $[x, p_x^3] = 3i\hbar p_x^2$ [5]

(iii) $[L_+, L_-] = 2\hbar L_z$ [5]

where $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$

(B) A particle is described by the wave function

$$\psi(x) = \left(\frac{\pi}{a}\right)^{-1/4} \quad \exp(-a x^2/2)$$

Show that $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{2a}}$. [12]

Q.5.(A) Radial part of the Schrodinger equation for spherically symmetric potentials for $l = 0$ is given by the equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{2m}{\hbar^2}[E - V(r)]R = 0$$

(i) Show that with $u(r)=r\,R(r)$, the above equation reduces to [5]

$$\frac{d^2u(r)}{dr^2}+\frac{2m}{\hbar^2}[E-V(r)]u(r)=0$$

What are the boundary conditions on $u(r)$ for bound states.

(ii) A spherical oscillator potential is given by

$$V(r)=\frac{1}{2}m\omega^2 r^2$$

(a) Show that the reduced radial equation is identical with one dimensional linear oscillator. [5]

(b) Assume that the lowest energy state is given by the wave-function $A\exp(-\frac{1}{2}\xi^2)$ where $\xi=\sqrt{\frac{m\omega}{\hbar}}\,r$. [5]

Determine the energy E.

(B) The Hamiltonian of a system with moment of inertia I is given by the expression

$$H=\frac{1}{2I_1}(L_x^2+L_y^2)+\frac{1}{2I_3}L_z^2$$

(i) Show that $[H, L^2]=0$ and $[H, L_z]=0$. [5]

(ii) Find an expression for the eigenvalue of the Hamiltonian. [5]
Eigen functions are $Y_l^m(\vartheta,\varphi)$.
Here L is orbital angular momentum.

@@@@END OF EXAMINATION@@@@

APPENDIX:

Given: $\hbar=1.0546\times10^{-34}$ J s , *c = velocity of light* $=2.99792\times10^{8}$ m s^{-1}
mass of neutron/proton $=1.6749\times10^{-27}$ kg , $k=1.3807\times10^{-23}$ JK^{-1}.

1ev=1.6022×10^{-19} *J.*

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

$[r_i, p_j]=i\hbar\,\delta_{ij}$ where $r_i=(x, y, z)$ and $p_i=(p_x, p_y, p_z)$,
$[L_x, L_y]=i\hbar L_z$, $[L_y, L_z]=i\hbar L_x$, $[L_z, L_x]=i\hbar L_y$ where $\vec{L}=\vec{r}\times\vec{p}$,

The functions $Y_l^m(\vartheta,\varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2 Y_\ell^m(\vartheta,\varphi)=\ell(\ell+1)\hbar^2 Y_\ell^m(\vartheta,\varphi)$$

$$L_z Y_\ell^m(\vartheta,\varphi)=m\hbar Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_0^{\infty} exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} t^{2n+1} exp(-at^2)\,dt = \frac{n!}{2a^{n+1}} \qquad \text{with Re } a > 0, \; n = 0,1,2,..$$

$$\int_0^{\infty} t^{2n} exp(-at^2)\,dt = \frac{1.3.5......(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....

$$\int sin^2(x)dx = \frac{x}{2} - \frac{1}{4}sin(2x)$$

$$\int sin(mx)sin(nx)dx = \frac{1}{2}\left[\frac{sin\{(m-n)x\}}{(m-n)} - \frac{sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int sin(mx)cos(nx)dx = -\frac{1}{2}\left[\frac{cos[(m-n)x]}{(m-n)} + \frac{cos[(m+n)x]}{(m+n)}\right]$$

$$\int_{-\infty}^{\infty} H_n(\xi)\,H_m(\xi)\,exp(-\xi^2)d\xi = \pi^{\frac{1}{2}}\,2^n\,n!\,\delta_{nm}$$

where H(ξ) are Hermite polynomials and are real.

$$\int_0^{\infty} t^{z-1}\exp(-kt)\,dt = k^{-z}\,\Gamma(z) \quad \text{Re}\,z > 0, \text{Re}\,k > 0.$$

$$\Gamma(n+1) = n! \quad \text{for } n = 1,2,\ldots \text{ and } \Gamma(1) = 1.$$

$$\int x^n e^{-m\alpha x}\,dx = \frac{\Gamma(n+1)}{(m\alpha)^{n+1}} \qquad \text{for } m > 0 \quad \text{and } n \geq 0.$$

You can calculate the integrals you need by expressing powers of x through (repeated) differentiation with respect to the parameter in the exponential, e.g.

$$\int_a^b dx\,x\exp(-\gamma x) = -\frac{\partial}{\partial\gamma}\int_a^b dx\exp(-\gamma x) \;\; \text{and} \;\; \int_a^b dx\,x^2\exp(-\gamma x) = \frac{\partial^2}{\partial\gamma^2}\int_a^b dx\exp(-\gamma x)$$ *and so on.).*