UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2007/2008

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) Given P(35 , 230^0 , - 7) in a cylindrical coordinate system, find its Cartesian and spherical coordinates . **(5 marks)**

(b) Given a scalar function f in Cartesian system as $f = x^2 + 6y$,

(i) find $\vec{\nabla} f$ at the point (x = 1 , y = 6 , z = 0) , **(3 marks)**

(ii) find the directional derivative at the same point along the direction of $\vec{e}_x \, 3 + \vec{e}_y \, 4$. **(3 marks)**

(c) Given $\vec{F} = \vec{e}_x \, y^2 + \vec{e}_y \, 5xy + \vec{e}_z \, 4z$, evaluate the value of the line integral $\int_L \vec{F} \bullet d\vec{l}$

(i) if L : the straight line segment from $(0 , 0)$ to $(1 , 1)$ on $z = 0$ plane, **(5 marks)**

(ii) if L is a parabolic line segment of $y = x^2$ from $(0 , 0)$ to $(1 , 1)$ on $z = 0$ plane, **(6 marks)**

(iii) find $\vec{\nabla} \times \vec{F}$ and comment on whether or not the given $\vec{F}$ is a conservative vector field . **(3 marks)**

Question two

Given $\vec{F} = \vec{e}_r \, r^3 + \vec{e}_\theta \, r^3 \sin(\phi) + \vec{e}_\phi \, r^3 \cos(\theta)$,

(a) evaluate the value of the closed surface integral $\oiint_S \vec{F} \bullet d\vec{s}$ if S : the closed surface of a sphere of a radius 5 and centred at the origin , i.e.,

$d\vec{s} = \vec{e}_r \, r^2 \sin(\theta) \, d\theta \, d\phi$ with $r = \mathbf{5}$, **(9 marks)**

(b) (i) find $\vec{\nabla} \bullet \vec{F}$, **(5 marks)**

(ii) find the value of the volume integral $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is the volume enclosed by the given closed surface in (a) , i.e.,

$dv = r^2 \sin(\theta) \, dr \, d\theta \, d\phi$, $\mathbf{0} \leq r \leq \mathbf{5}$, $\mathbf{0} \leq \theta \leq \pi$ and $\mathbf{0} \leq \phi \leq \mathbf{2}\pi$. Compare the result here with that obtained in (a) and make brief comment about the Divergence Theorem. **(11 marks)**

Question three

(a) Given the following partial differential equation as

$$\frac{\partial^2 f(\rho,\phi)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f(\rho,\phi)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f(\rho,\phi)}{\partial \phi^2} = 0 .$$

Set $f(\rho,\phi) = F(\rho)\, G(\phi)$ and use the separation of variables scheme to break the given partial differential equation into two ordinary differential equations .**(6 marks)**

(b) Given the following differential equation as :

$$(1-x^2)\frac{d^2 y(x)}{dx^2} - 2x\frac{dy(x)}{dx} + 30\, y(x) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ and $a_0 \neq 0$,

(i) write down the indicial equations and find the values of s and also possibly the value of a_1 , **(6 marks)**

(ii) write down the recurrence relation. For each possible values of s found in (b) (i) , set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_5 , **(11 marks)**

(iii) write down the general solution of the given differential equation. **(2 marks)**

Question four

(a) An elastic string of length **8** is fixed at its two ends, i.e., at $x = 0$ and $x = 8$.

Given one-dimensional wave equation governing the transverse deflection of the string $u(x,t)$ as :

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\frac{\partial^2 u(x,t)}{\partial x^2}$$

(i) show that a solution of this form $u(x,t) = \sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{8}) \cos(\frac{n\pi t}{4})$ where E_n is an arbitrary constant satisfying two fixed end conditions as well as zero initial speed condition , **(6 marks)**

(ii) if the initial position of the string , i.e., $u(x,0)$, is given as

$$u(x,0) = \begin{cases} \frac{5}{3}x & \text{if } \quad 0 \le x \le 3 \\ -x+8 & \text{if } \quad 3 \le x \le 8 \end{cases}$$

find E_n in terms of n and calculate the values of E_1 , E_2 *and* E_3 .

(hint : $\int_{x=0}^{L} \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L})\, dx = \begin{cases} 0 & \text{if } n \ne m \\ \frac{L}{2} & \text{if } n = m \end{cases}$) **(10 marks)**

(b) Given a non-homogeneous differential equation as

$\frac{d^2 x(t)}{dt^2} - 2\frac{d\,x(t)}{dt} + 3\,x(t) = 5\cos(3t)$, set the particular solution of x(t) as

$k_1 \cos(3t) + k_2 \sin(3t)$ and determine the values of k_1 and k_2 . **(9 marks)**

Question five

Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2x_1(t)}{dt^2} = -11\ x_1(t) + 9\ x_2(t) \\ \dfrac{d^2x_2(t)}{dt^2} = 3\ x_1(t) - 5\ x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ *and* $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\,X = -\,\omega^2\,X$ where

$$A = \begin{pmatrix} -11 & 9 \\ 3 & -5 \end{pmatrix} \quad \textit{and} \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(3 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(4 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (a), **(5 marks)**

(d) (i) write down the general solutions for $x_1(t)$ and $x_2(t)$, **(3 marks)**

(ii) if initially their positions are $x_1(0) = 0$ and $x_2(0) = 3$ and their speeds are zero , i.e., $\left.\dfrac{d\,x_1(t)}{d\,t}\right|_{t=0} = \left.\dfrac{d\,x_2(t)}{d\,t}\right|_{t=0} = 0$, determine the values of the arbitrary constants in (c) (i) and thus find the specific solution. **(10 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin s\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3} \right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3} \right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1} \right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2} \right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3) represents (x, y, z) for rectangular coordinate system
represents (ρ, ϕ, z) for cylindrical coordinate system
represents (r, θ, ϕ) for spherical coordinate system

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ represents $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ for rectangular coordinate system
represents $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$ for cylindrical coordinate system
represents $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ for spherical coordinate system

(h_1, h_2, h_3) represents $(1, 1, 1)$ for rectangular coordinate system
represents $(1, \rho, 1)$ for cylindrical coordinate system
represents $(1, r, r\sin\theta)$ for spherical coordinate system