UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2007/2008

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **SIX** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1. (a) An atom can radiate any time after it is excited. It is found that in a typical case, the average excited atom has life time of the order of 10^{-8} s . That is, during this period it emits a photon and is de-excited. Estimate the minimum uncertainty $\Delta\nu$ in the frequency of photon. [5]

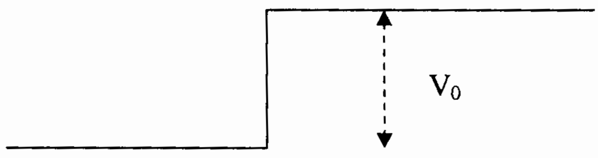
(b) Nuclei, typically of size 10^{-14} m , frequently emit electrons, with typical energies in the range of 1 to 10 MeV. Use the uncertainty principle to show that electrons of energy 1 MeV could not be contained in the nucleus before decay. [5]

(c) Find the explicit expression for the following operator [5]

$$\Omega = \left(-i\hbar \frac{d}{dx} + x \right)^2$$

Is the operator Ω a Hermitian operator?

(d) Consider a one dimensional step potential of the type [10]



For a particle of mass m and energy $E < V_0$, derive an expression for the current density for $x<0$ and $x>0$.

Note: Current density in one dimension is given by the expression

$$S(x,t) = \frac{\hbar}{2im}\left(\psi^*(x,t)\frac{d\psi(x,t)}{dx} - \frac{d\psi^*(x,t)}{dx}\psi(x,t) \right)$$

Q.2. (a) Show that the wave function $\psi(x,t)$ with energy E for a time independent potential can written as [5]

$$\psi(x,t) = u(x)\exp(-iEt/\hbar)$$

where $u(x)$ is the solution of time independent Schrodinger equation.

(b) Consider a particle of mass m which can move along the x-axis from x=-L/2 to x=L/2 but which is strictly prohibited from being found outside this region. The wave function for the lowest energy state of the particle is

$$\psi(x,t) = A\cos(\pi x/L)\exp(-iEt/\hbar) \quad \text{for} \quad -L/2 \le x \le L/2$$

where A is a real constant, and E is the total energy of the particle.

(i) Show that it is a solution of the Schrodinger equation with specific conditions on the potential assumed. Indicate the conditions on the potential. Determine the expression for the energy E. [10]

(ii) Sketch the potential in whch the particle moves. [2]

(iii) Evaluate the constant A. [2]

(iv) Evaluate the expectation value of x and p_x of the particle associated with the wave function. [6]

Q.3. (a) A Hamiltonian H has two eigen-functions ψ_0 and ψ_1 belonging to two different energies E_0 and E_1 respectively. Show that the two eigen-functions are orthogonal. [5]

(b) If the two eigen-functions belong to the same energy, show that they need not be orthogonal. [3]

(b) Show that the two wave-functions

$$\psi_0 = \left(\frac{a}{\pi}\right)^{1/4} \exp(-ax^2/2) \quad ; \quad \psi_1 = \left(\frac{a}{\pi}\right)^{1/4} (2a)^{1/2} \quad x\exp(-ax^2/2)$$

are eigen functions of the Hamiltonian

$$H = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2} + a^2x^2\right)$$

belonging to two different energies. [5]

(i) Show that the two eigen functions are ortho-normal. [4]

(ii) What are the parities of the two wave functions. [2]

(iii) Show that $\Delta x = \sqrt{<x^2> - <x>^2} = (1/2a)^{1/2}$. [8]
Here $<>$ corresponds to expectation value.

(iv) Sketch the potential of this problem. [1]

(v) In the same figure of (iv) sketch the probability of finding the particle in state ψ_0 and ψ_1. [2]

Q.4. (a) Let α and β be two Hermitian operators whose eigen-vectors are $|ab\rangle$ with the property

$$\alpha|ab\rangle = a|ab\rangle$$
$$\beta|ab\rangle = b|ab\rangle$$

where a and b are corresponding eigen-values.
Show that $[\alpha, \beta] = 0$. [5]

(b) Using the relation $[x, p_x] = i\hbar$, show that [5]

$$[x, p_x^3] = 3i\hbar p_x^2$$

(c) Using the relations $[\sigma_x, \sigma_y] = i\sigma_z$, $[\sigma_y, \sigma_z] = i\sigma_x$, *and* $[\sigma_z, \sigma_x] = i\sigma_y$ show that

(i) $[\sigma_i, \sigma^2] = 0$ for i=x,y,z and $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$. [5]

(ii) $[\sigma_+, \sigma_-] = 2i\sigma_z$ [5]
$[\sigma_z, \sigma_+] = \sigma_+$
$[\sigma_z, \sigma_-] = -\sigma_-$
where $\sigma_+ = \sigma_x + i\sigma_y$ and $\sigma_- = \sigma_x - i\sigma_y$.

(iii) if $|\alpha\beta\rangle$ are eigen vectors of σ^2 of eigen value α and σ_z with eigen value β then σ_+ raises the value of β by 1 and σ_- lowers the value of β by 1. [5]

Q.5. (a) Radial part of the Schrodinger equation for spherically symmetric potentials for $l = 0$ is given by the equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2m}{\hbar^2}[E-V(r)]\,R=0$$

(i) Show that with $u(r)=r\,R(r)$, the above equation reduces to [5]

$$\frac{d^2u(r)}{dr^2}+\frac{2m}{\hbar^2}[E-V(r)]u(r)=0$$

What are the boundary conditions on $u(r)$ for bound states.

(ii) A spherical oscillator potential is given by

$$V(r)=\frac{1}{2}m\omega^2 r^2$$

(a) Show that the reduced radial equation is identical with one dimensional linear oscillator. [5]

(b) Assume that the lowest energy state is given by the wave-function $A\exp(-\frac{1}{2}\xi^2)$ where $\xi=\sqrt{\frac{m\omega}{\hbar}}\,r$. [5]

Determine the energy E.

(b) An electron is described by an Hamiltonian $H = H_0 + H_1$ where $H_0=\frac{p^2}{2m}-\frac{e^2}{r}$ and H_1 describes the contribution from external force.

H_0 has the eigen functions $\psi_{nlm}=R_{nl}(r)\,Y_l^m(\vartheta,\varphi)$ and energy E_n.

Here

n = principal quantum number,

l = angular momentum quantum number

m = projection of angular momentum.

The eigen functions $\psi_{n\ell m}$ have following properties:

$$H_0\,\psi_{n\ell m} = E_n\,\psi_{n\ell m}$$

$$H_1\,\psi_{n\ell m} = \alpha\,\psi_{n\ell m}$$

A state of the electron is described by eigen-function $\phi = N(\psi_{100}+\sqrt{3}\psi_{200}-\sqrt{2}\psi_{210})$ and the eigen energy for the electron is given by $H\varphi = E\varphi$.

(i) Determine the normalization constant N. [5]

(ii) Determine the expectation value of H. [5]

Note: $\int \psi^*_{n_1\ell_1m_1}\,\psi_{n_2\ell_2m_2}\,d\tau=\delta_{n_1n_2}\,\delta_{\ell_1\ell_2}\,\delta_{m_1m_2}$

@@@@END OF EXAMINATION@@@@

Appendix:

PHYSICAL CONSTANTS AND DERIVED QUANTITIES

Speed of light c = 2.99792458 x 10^8 m s^{-1} ~ 3.00 x 10^{23} fm s^{-1}
Avogadro's number N_A = 6.02214199(47) x 10^{26} molecules per kg-mole
Planck's constant h = 6.626068 76(52) x 10^{-34} J s

$\hbar$ = 1.054571 596(82) x 10^{-34} J s = 0.65821 X 10^{-21} MeV s

$\hbar^2$ = 41.802 u MeV fm^2

$\hbar$ c = 197.327 MeV fm

Fermi 1fm = 10^{-15} m
1 eV=1.6022 x 10^{-9} J 1 MeV=1.602176 x 10^{-13} J

Elementary charge e = 1.602176462(63) x 10^{-19}C
$e^2/4\pi\varepsilon_0$ = 1.4400MeV fm

Fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$ = 1 / 137.036

Boltzmann constant k = 1.3806503(24) x 10^{-23} JK^{-1} = 0.8617 x 10^{-4} eV K^{-1}

MASSES AND ENERGIES
Atomic mass unit m_u or u = 1.66053873(13) x 10^{-27} kg
$m_u c^2$ = 931.494 MeV

Electron	m_e	= 9.10938188(72) x 10^{-31} kg
	m_e/m_u	= 5.486 x 10^{-4} = 1/1823
	$m_e c^2$	= 0.510998902(21) MeV
Proton	m_p	= 1.67262158(13) x 10^{-27} kg
	m_p / m_u	= 1.00727647
	$m_p c^2$	= 938.272 MeV
Hydrogen atom	m_H	= 1.673533 x 10^{-27} kg
	m_H / m_u	= 1.007825
	$m_H c^2$	= 938.783 MeV
Neutron	m_n	= 1.67492716(13) x 10^{-27} kg
	m_n/m_u	= 1.00866491578(55)
	$m_n c^2$	= 939.565 MeV
Alpha particle	m_α	= 6.644656 x 10^{-27} kg
	m_α /m_u	= 4.001506175
	$m_\alpha c^2$	= 3727.379 MeV

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2 Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\vartheta,\varphi)$$

$$L_z Y_\ell^m(\vartheta,\varphi) = m\hbar Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_{-a}^{+a} \cos^2(kx)dx = \frac{\cos(ka)\sin(ka)+ka}{k} \quad ; \quad \int_{-a}^{+a} \cos(kx)\sin(kx)dx = 0 \quad ; \quad \int_{-a}^{+a} x\cos^2(kx)dx = 0$$

$$\int_{-a}^{+a} x\cos(kx)\sin(kx)dx = \frac{2ka\cos^2(ka) - \cos(ka)\sin(ka) - ka}{2k^2} \; .$$

$$\int_0^\infty exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1} exp(-at^2)\,dt = \frac{n!}{2a^{n+1}}$$ with Re a > 0, n =0,1,2,..

$$\int_0^\infty t^{2n} exp(-at^2)\,dt = \frac{1.3.5......(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....