UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2007/2008

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **FIVE** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1.

(a) (i) Calculate the de-Braglie wavelength for an electron whose velocity is 10^7 m/sec. [2]

(ii) A bullet weighing $5x10^{-3}$ kg takes 0.5 sec to reach its target. Regarding the Bullet as a mass point, and neglecting effects of air resistance and earth motion, find the order of magnitude of the spread of successive shots at the target under optimum conditions of aiming and firing. Use the uncertainty principle $\Delta x \Delta p \geq \hbar$. [3]

(b) Write short notes on

(i) Parity [2]

(ii) Complete set of states. [2]

(c) For each of the following decide whether they have even, odd or definite parity. [4]

(i) $x\exp(-\alpha x^2)$ where α is a real constant.

(ii) $A\exp(ikx)+B\exp(-ikx)$ where k is a real constant, A and B are complex constants.

(d) General time dependent Schrodinger equation is [4]

$$i\hbar\frac{d\psi(x,t)}{dt}=-\frac{\hbar^2}{2m}\frac{d^2\psi(x,t)}{dt^2}+V(x,t)\psi(x,t)=E\psi(x,t)$$

For the potential V(x,t)=V(x), that is, potential does not depend on time, show that

$$\psi(x,t)=u(x)\exp(-iEt/\hbar)$$

where E=energy of the particle of mass m, and u(x) is the solution of the time independent Schrodinger equation.

(e) The wave function of a particle moving in one dimension is give by: [8]

$$\psi(x)=0 \quad \text{for } x<0$$
$$=B\sqrt{x}\exp(-\beta x) \quad \text{for } x\geq 0$$

where β is real and positive constant. Note x varies from -∞ to +∞ .

(i) Calculate the normalization constant B. (It is a function of β.)

(ii) Calculate the average position of the particle on the x-axis, as a function of β

Q.2. The two wave functions

$$\varphi_0(x)=\left(\frac{a}{\pi}\right)^{1/4}\exp(-a x^2/2)$$

and

$$\varphi_1(x)=\left(\frac{a}{\pi}\right)^{1/4}(2a)^{1/2}\, x\exp(-a x^2/2)$$

are solutions of the eigenvalue problem

$H\varphi_n(x)=E_n\varphi_n(x)$ where n=0,1,2,........

(i) Show that $\varphi_0(x)$ and $\varphi_1(x)$ are normalized. [4]

(ii) Show that $\varphi_0(x)$ and $\varphi_1(x)$ are orthogonal. [2]

(iii) Show that $\varphi_0(x)$ is an eigen function if $V(x)=a^2x^2$ of the Hamiltonian [9]

$$H=-\frac{d^2}{dx^2}+V(x)$$

(iv) Determine the eigen value for the eigen functions $\varphi_0(x)$ and $\varphi_1(x)$. [4]

(v) Show that $\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=(1/2a)^{1/2}$ for $\varphi_0(x)$. [6]

Note: The symbol $\langle\ \rangle$ corresponds to expectation value.

$\int_{-\infty}^{\infty} f(x)dx=0$ if *f(x)* is odd function of x.

Q.3. $\phi(x)$ is any of the odd-parity solutions to the stationary Schrödinger equation for a finite square well

$$V(x)=\begin{cases} -V_0 & \text{for } -\frac{L}{2}<x<\frac{L}{2} \\ 0 & \text{for } |x|>\frac{L}{2} \end{cases}$$

(i) Sketch the potential. [2]
(ii) State the boundary conditions for bound states. [2]
(iii) Show that the solutions for the bound states ($E<0$) can be written as [10]

$$\phi(x)=\begin{cases} F\exp(-k\,x) & \text{for } x>L/2 \\ B\sin(q\,x) & \text{for } -\frac{L}{2}\le x\le\frac{L}{2} \\ -F\exp(k\,x) & \text{for } x<0, \text{ and } |x|<L/2 \end{cases}$$

where *F* and *B* are constants, $k\equiv\sqrt{-2mE}/\hbar$, and $q\equiv\sqrt{2m(E+V_0)}/\hbar$.

(iv) State the continuity conditions on $\phi(x)$. [2]

(v) Using the continuity on $\phi(x)$ and $d\phi/dx$, show that [9]

$$-k=-q\cot(qL/2)$$

Q.4.

(a) Explain the following:

(i) What is the difference between a state of the system given by the ket $|\ell m\rangle$ and the wave function $\varphi_{\ell m}(r,\vartheta,\varphi)$ describing the same state. [2]

(ii) A dynamical quantity is always represented by a Hermitian linear operator. [2]

(b) Show that the eigen-functions of a Hermitian operator corresponding to two different eigenvalues are orthogonal. [10]

(c) Given that $[\ r_i\ ,\ p_j\ ] = i\hbar\ \delta_{ij}$ where $r_i = (\ x\ ,\ y\ ,\ z\ )$ and $p_i = (\ p_x\ ,\ p_y\ ,\ p_z\ )$, and $[\ L_x\ ,\ L_y\ ] = i\hbar\ L_z$ where $\vec{L}=\vec{r}\times\vec{p}$, show that

(i) $[\ x\ ,\ p^2\ ] = 2\ i\hbar\ p_x$ where $p^2 = p_x^2 + p_y^2 + p_z^2$. [6]

(ii) $[L_+\ ,L_-\ ] = 2\hbar\ L_z$ [5]

where $L_+ = L_x + i\,L_y$ and $L_- = L_x - i\,L_y$

Q.5.

(a) The Hamiltonian of a rotating system with moment of inertia I is given by the expression

$$H=\frac{1}{2I}(\ L_x^2+L_y^2\ ) \qquad \text{where } \vec{L}=\vec{r}\times\vec{p}\ .$$

Show that $Y_\ell^m(\vartheta,\varphi)$ are eigen-functions of H . [10]

(b) An electron is described by an Hamiltonian $H = H_0 + H_1$ where $H_0 = \frac{p^2}{2m} - \frac{e^2}{r}$ and H_1 describes the contribution from external force.

H_0 has the eigen functions $\psi_{nlm} = R_{nl}(r)\,Y_l^m(\,\vartheta,\varphi\,)$ and energy E_n .

Here

n = principal quantum number,
l = angular momentum quantum number
m = projection of angular momentum.

The eigen functions $\psi_{n\ell m}$ have following properties:

$$H_0\,\psi_{n\ell m} = E_n\,\psi_{n\ell m}$$
$$H_1\,\psi_{n\ell m} = \alpha\,\psi_{n\ell m}$$

A state of the electron is described by eigen-function $\phi = N(\psi_{100} - \sqrt{2}\,\psi_{210})$ and the eigen energy for the electron is given by $H\varphi = E\varphi$.

(i) Determine the normalization constant N. [5]

(ii) Determine the expectation value of H . [10]

Note: $\int \psi^*_{n_1\ell_1m_1}\,\psi_{n_2\ell_2m_2}\,d\tau = \delta_{n_1n_2}\,\delta_{\ell_1\ell_2}\,\delta_{m_1m_2}$

@@@@END OF EXAMINATION@@@@

APPENDIX

$\hbar = 1.0546 \times 10^{-34}$ Js , *c = velocity of light = 2.99792 x 10^{8} m s^{-1}*
mass of neutron/proton = 1.6749 x 10^{-27} kg , k=1.3807x 10^{-23} JK^{-1} .

1ev=1.6022 x 10^{-19} J. , mass of electron= 9.10938 x 10^{-31} kg.

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

$[\, r_i , p_j \,] = i\hbar\ \delta_{ij}$ **where** $r_i = (x, y, z)$ **and** $p_i = (p_x, p_y, p_z)$,
$[\, L_x , L_y \,] = i\hbar\ L_z$, $[\, L_y , L_z \,] = i\hbar\ L_x$, $[\, L_z , L_x \,] = i\hbar\ L_y$ **where** $\vec{L} = \vec{r} \times \vec{p}$,

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2\, Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2\, Y_\ell^m(\vartheta,\varphi)$$
$$L_z\, Y_\ell^m(\vartheta,\varphi) = m\,\hbar\, Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$\Gamma(n+1) = n!$ *for* $n = 1,2,\ldots$ *and* $\Gamma(1) = 1.$

$$\int_0^\infty t^{z-1}\exp(-kt)\,dt = k^{-z}\,\Gamma(z) \quad \mathrm{Re}\,z > 0, \mathrm{Re}\,k > 0. \;,$$

$$\int x^n e^{-m\alpha x}\,dx = \frac{\Gamma(n+1)}{(m\alpha)^{n+1}} \qquad \text{for } m > 0 \text{ and } n \ge 0.$$

$$\int_{-\infty}^{\infty} dz\, e^{-\alpha z^2} = \sqrt{\frac{\pi}{\alpha}} \quad , \quad \int_{-\infty}^{\infty} dz\, z^2\, e^{-\alpha z^2} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \quad , \quad \int_{-\infty}^{\infty} dz\, z^4\, e^{-\alpha z^2} = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$

$$\int_0^\infty \exp(-t^2)\,dt = \frac{\sqrt{\pi}}{2} \quad , \quad \int_0^\infty t^{2n+1}\exp(-at^2)\,dt = \frac{n!}{2a^{n+1}} \qquad \text{with Re } a > 0,\ n = 0,1,2,..$$

$$\int_0^\infty t^{2n}\exp(-at^2)\,dt = \frac{1.3.5......(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} \qquad \text{with Re } a > 0,\ n=0,1,2.....$$

$$\int \sin^2(x)\,dx = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int \sin(mx)\sin(nx)\,dx = \frac{1}{2}\left[\frac{\sin\{(m-n)x\}}{(m-n)} - \frac{\sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int \sin(mx)\cos(nx)\,dx = -\frac{1}{2}\left[\frac{\cos[(m-n)x]}{(m-n)} + \frac{\cos[(m+n)x]}{(m+n)}\right]$$

You can calculate the integrals you need by expressing powers of x through (repeated) differentiation with respect to the parameter in the exponential, e.g.

$$\int_a^b dx\; x\exp(-\gamma x) = -\frac{\partial}{\partial\gamma}\int_a^b dx\exp(-\gamma x) \;\; \text{and} \;\; \int_a^b dx\; x^2\exp(-\gamma x) = \frac{\partial^2}{\partial\gamma^2}\int_a^b dx\exp(-\gamma x) \;\; \text{and so on.).}$$