UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2008/2009

TITLE OF PAPER : MATHEMATICAL METHODS FOR PHYSICISTS

COURSE NUMBER : P272

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF SIX QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) Given P(- 5 , - 3 , - 7) in Cartesian coordinate system, find its cylindrical and spherical coordinates . **(5 marks)**

(b) Given a scalar function f in Cartesian system as $f = 7\,x^2 - 5\,x\,y + z^2$,

find $\vec{\nabla} f$ at the point P (x = 2 , y = - 3 , z = 1) , **(4 marks)**

(c) Given a vector field $\vec{F} = \vec{e}_x\, 2\,x\,y + \vec{e}_y \left(x^2 - 12\,y\,z\right) - \vec{e}_z\, 6\,y^2$, find

the value of line integral of $\vec{F}$ from the point P_1 : (1 , 2 , 0) to the point

P_2 : (5 , 14 , 0) along a line path of L , i.e., $\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$,

(i) if L is a straight line from P_1 to P_2 , **(8 marks)**

(ii) if L is a parabolic path from P_1 to P_2 described by $y = \frac{1}{2}x^2 + \frac{3}{2}$

on $z = 0$ plane . Compare this result with that obtained in (c)(i) and make a brief comment on whether the given $\vec{F}$ is a conservative vector field or not?

(8 marks)

Question two

Given a vector field $\vec{F} = \vec{e}_r \, 5 \, r^2 - \vec{e}_\theta \, r^2 + \vec{e}_\theta \, r^2 \sin(\phi)$

(a) evaluate the value of the closed surface integral $\oiint_S \vec{F} \bullet d\vec{s}$ if S the closed surface of a sphere of a radius 4 and centred at the origin, i.e.,

$$d\vec{s} = \vec{e}_r \, r^2 \sin(\theta) \, d\theta \, d\phi \xrightarrow{r=4} \vec{e}_r \, 16 \sin(\theta) \, d\theta \, d\phi \quad , \quad 0 \le \theta \le \pi \quad \& \quad 0 \le \phi \le 2\pi$$

(9 marks)

(b) (i) find $\vec{\nabla} \bullet \vec{F}$, **(5 marks)**

(ii) find the value of the volume integral $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is the volume enclosed by the given closed surface in (a) , i.e.,

$$dv = r^2 \sin(\theta) \, dr \, d\theta \, d\phi \quad , \quad 0 \le r \le 4 \quad , \quad 0 \le \theta \le \pi \quad \& \quad 0 \le \phi \le 2\pi$$

Compare the result here with that obtained in (a) and make brief comment about the Divergence Theorem. **(11 marks)**

Question three

Given the following differential equation as :

$$\frac{d^2 y(x)}{d x^2} - \frac{d y(x)}{d x} + 3\, y(x) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$,

(a) write down the indicial equations. Find the values of s and a_1 (by setting $a_0 = 1$) .

(10 marks)

(b) write down the recurrence relation. For all the appropriate values of s and a_1 found in (a), set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_5 . Thus write down two independent solution in their polynomial forms.

(15 marks)

Question four

An elastic string of length 10 is fixed at its two ends, i.e., at $x = 0$ & $x = 10$ and its transverse deflection $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\,\frac{\partial^2 u(x,t)}{\partial x^2} \quad ,$$

(a) use separation of variable scheme to split the above partial differential equation into two ordinary differential equations and then write down the general solution of $u(x,t)$.

(8 marks)

(b) given the general solution of $u(x,t)$, after satisfying two fixed end conditions as well as zero initial speed condition, as $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right)\cos\left(\frac{3n\pi t}{10}\right)$ where

E_n $n = 1,2,3,\cdots\cdots$ are arbitrary constants , then find E_n in terms of n and calculate the values of E_1 , E_2 & E_3 if the initial position of the string , i.e., $u(x,0)$, is given as

$$u(x,0) = \begin{cases} 4x & \text{if } 0 \le x \le 2 \\ -x + 10 & \text{if } 2 \le x \le 10 \end{cases}$$

(hint : $\int_{x=0}^{10} \sin\left(\frac{n\pi x}{10}\right)\sin\left(\frac{m\pi x}{10}\right) dx = \begin{cases} 0 & \text{if } n \ne m \\ 5 & \text{if } n = m \end{cases}$ &

$\int x \sin\left(\frac{n\pi x}{10}\right) dx = \frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right) - \frac{10}{n\pi}\, x\cos\left(\frac{n\pi x}{10}\right)$) **(17 marks)**

Question five

Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -5\, x_1(t) + 16\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 4\, x_1(t) - 17\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\, X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -5 & 16 \\ 4 & -17 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(5 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(6 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b), **(8 marks)**

(d) find the normal coordinates of the given coupled system. **(6 marks)**

Question six

Given the following non-homogeneous differential equation as :

$$\frac{d^2 x(t)}{dt^2} + 2\frac{d\,x(t)}{dt} + 5\,x(t) = 8\,e^{-t} + 26\sin(3t)$$

(a) set its particular solution as $x_p(t) = k_1\,e^{-t} + k_2\,\sin(3t) + k_3\,\cos(3t)$, determine the values of k_1 , k_2 & k_3 . **(9 marks)**

(b) find the general solution to the homogeneous part of the given differential equation , i.e.,

$\frac{d^2 x(t)}{dt^2} + 2\frac{d\,x(t)}{dt} + 5\,x(t) = 0$, and name it as $x_h(t)$. **(6 marks)**

(c) write the general solution to the given non-homogeneous differential equation as $x(t) = x_p(t) + x_h(t)$ and determine the values of the arbitrary constants in this general solution if the initial conditions are given as $x(0) = 5$ & $\left.\frac{d\,x(t)}{dt}\right|_{t=0} = 0$.

(10 marks)

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system