UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2008-09

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **Six** PAGES, INCLUDING THIS PAGE.

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Q.1.

(a) The average lifetime of an excited state of an atom is about 10^{-8} sec. Using this as Δt for the emission of a photon, compute the minimum $\Delta \nu$ permitted by the uncertainty principle. [5]

(b) Electromagnetic radiation consists a collection of quanta known as photons with energy $h\nu$. Calculate the number of photons emitted by **100** watt source with $\lambda = 600 \times 10^{-9}$ **m** . [5]

(c) Consider a free particle of mass m and momentum p in one dimension.

(i) Write down the Hamiltonian. [1]

(ii) Write down the time dependent Schrodinger equation for the given problem. [1]

(iii) Show that [6]

$$\psi_k(x,t) = [A\exp(ikx) + B\exp(-ikx)]\exp(-iE_k t/\hbar)$$

is a solution of the time-dependent Schrodinger equation in one dimension and that

$$E_k = \frac{\hbar^2 k^2}{2m}$$

(iv) Show that $\psi^*(x,t)\psi(x,t)$ is real. [2]

(v) Find the relation between k and **p**. [1]

(vi) Show that the probability current $S(x,t)$ corresponding to $\psi_k(x,t)$ is given by [4]

$$S(x,t) = \frac{\hbar k}{m}\left[|A|^2 - |B|^2\right]$$

What is the interpretation of this?

Note: *Current density in one dimension is given by the expression*

$$S(x,t) = \frac{\hbar}{2im}\left(\psi^*(x,t)\frac{d\psi(x,t)}{dx} - \frac{d\psi^*(x,t)}{dx}\psi(x,t)\right)$$

Q.2.

(a) Explain the concept of parity. [1]

For each of the following functions, decide whether they have even, odd or no definite parity:

(i) $x^2 \sin(kx)$ where k is a constant. [1]

(ii) $A + Bx$. [1]

(b) A particle of mass m is bound in a one dimensional potential of the form

$$V(x) = -V_0 \text{ for } -L < x < L$$
$$= 0 \quad \text{for all other values of } x \text{ and } -\infty \le x \le \infty$$

(i) Write down the Schrodinger equations. [2]

(ii) Show that odd parity solutions for the bound states are [5]

$$\begin{aligned}\phi(x) &= A\exp(-kx) \quad for \quad x > L \\ &= B\sin(Kx) \quad for \quad -L<x<L \\ &= C\exp(kx) \quad for\ x<0 \quad and \quad |x| > L\end{aligned}$$

where $k^2 = \dfrac{2m|E|}{\hbar^2}$ and $K^2 = \dfrac{2m(V_0 - |E|)}{\hbar^2}$.

Here A , B and C are constants.

(iii) State the continuity conditions on $\phi(x)$ and $\dfrac{d\phi(x)}{dx}$ [2]

(iv) Use the continuity conditions to derive the following relations [8]

$$y = -z\cot z$$

$$y^2 + z^2 = R^2$$

$$where\ \ z = KL\ ,\quad y = kL \quad and \quad R^2 = \frac{2mV_0 L^2}{\hbar^2}$$

(v) Explain how bound state energies can be determined for a given value $\boldsymbol{V_0}$ and $\boldsymbol{L}$. [3]

(vi) Write down the normalization condition on $\phi(x)$ in terms of the solutions [2]
given above in (ii). ***Do not evaluate the integrals.***

Q.3.

(a) Explain the following:

(i) What is the difference between a state of the system given by the ket $|\ell m\rangle$ [2]
and the wave function $\varphi_{\ell m}(r,\vartheta,\varphi)$ describing the same state.

(ii) A dynamical quantity is always represented by a Hermitian linear operator. [2]

(b) Orbital angular momentum $\vec{L}$ is defined by $\vec{L}=\vec{r}\times\vec{p}$. Using the identity $[r_i , p_j] = i\hbar\,\delta_{ij}$ where ***i*** and ***j*** take the value ***1,2*** and ***3*** corresponding to ***x,y,z*** components respectively, show that

(i) $[L_x, x] = 0,\quad [L_x, y] = i\hbar z \quad and \quad [L_x, z] = -i\hbar y$ [6]

(ii) $[L_x, r^2] = 0$ where $r^2 = x^2 + y^2 + z^2$. [4]

(c) A Hamiltonian ***H*** is defined in terms of operators ***A*** and $\boldsymbol{A^\dagger}$ by

$$\boldsymbol{H = \omega A^\dagger A + \tfrac{1}{2}\hbar\,\omega}$$

and $\boldsymbol{Hu_E = E\,u_E}$ where ***E*** is energy of the system defined by ***H***. Given the property $\boldsymbol{[A, A^\dagger] = \hbar}$ and a function $\boldsymbol{v_E = A^+ u_E}$, show that

(i) $[H, A^{+}] = \hbar\omega A^{+}$ [5]

(ii) Use the result of (i) to show that v_E belongs to energy $E + \hbar\omega$. [6]

Q.4.

(a) A particle is described by the wave function

$$\psi(x) = \left(\frac{\pi}{a}\right)^{-1/4} \exp(-a x^2/2) \qquad \text{with} \quad -\infty < x < \infty$$

(i) Show that $\psi(x)$ is normalized. [2]

(ii) Show that $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{2a}}$ [8]

where the symbol $\langle\ \rangle$ corresponds to expectation value.

(iii) Show that $\Delta p_x = \sqrt{\langle p_x^2\rangle - \langle p_x\rangle^2} = \hbar\sqrt{\frac{a}{2}}$ where $p_x = -i\hbar\frac{\partial}{\partial x}$. [5]

(iv) What is the implication of the value of product $\Delta x\,\Delta p_x$? [2]

(b) The potential energy of a 3-dimensional harmonic oscillator is given by

$$V = \tfrac{1}{2}m[\omega_1^2 x^2 + \omega_2^2(y^2 + z^2)]$$

The energy eigen-values are given by the relation

$$E = (n_1 + \tfrac{1}{2})\hbar\omega_1 + (n_2 + n_3 + 1)\hbar\omega_2$$

where $n_1, n_2, n_3 = 0, 1, 2, \ldots$

(i) Determine E for its ground state. [1]
(ii) Determine E for its first excited state given by $n_1 = 0$. [4]
(iii) How many states belong to 1st excited state. Are they degenerate? [3]

Q.5.

(a) The Hamiltonian of a system is given by the expression [12]

$$H = \frac{1}{2I_1}(L_x^2 + L_y^2) + \frac{1}{2I_3}L_z^2$$

Find an expression for the eigenvalue of the Hamiltonian.
Here L is orbital angular momentum.

(b) An electron in the Coulomb field of a proton with Hamiltonian $H = H_0 + H_1$ is in a state described by the wave function

$$\phi = \frac{1}{6}\left[4\psi_{100} + 3\psi_{211} - \psi_{210} + \sqrt{10}\,\psi_{21-1}\right]$$

(i) Verify if the the wave function ϕ is normalized. [1]
(ii) Determine the expectation value of energy E, L^2 and L_z. [12]

Note that $\psi_{nlm} = R_{nl}(r)\,Y_l^m(\vartheta, \varphi)$ is the eigen function of H_0 with energy $E_n = \frac{E_0}{n^2}$

where E_0 is a constant and

n = principal quantum number= 1,2,3,.....,

l = angular momentum quantum number

m = projection of angular momentum onto the z-axis..

and $\int \psi^*_{n'l'm'} \psi_{nlm} \, d\tau = \delta_{n'n} \delta_{l'l} \delta_{m'm}$.

The eigen functions ψ_{nlm} have following properties:

$$H_0 \psi_{nlm} = E_n \psi_{nlm}$$

$$H_1 \psi_{nlm} = \alpha \psi_{nlm}$$

and the eigen energy for the electron is given by $H\phi = E\phi$.

@@@@END OF EXAMINATION@@@@

Appendix:

PHYSICAL CONSTANTS AND DERIVED QUANTITIES

1 Watt= 1 J s^{-1}
Speed of light c = 2.99792458 x 10^8 m s^{-1} ~ 3.00 x 10^{23} fm s^{-1}
Avogadro's number N_A = 6.02214199(47) x 10^{26} molecules per kg-mole
Planck's constant h = 6.626068 76(52) x 10^{-34} J s
$\hbar$ = 1.054571 596(82) x 10^{-34} J s = 0.65821 X 10^{-21} MeV s
$\hbar^2$ = 41.802 u MeV fm^2
$\hbar$ c = 197.327 MeV fm
Fermi 1fm = 10^{-15} m
1 eV=1.6022 x 10^{-9} J 1 MeV=1.602176 x 10^{-13} J

Elementary charge e = 1.602176462(63) x 10^{-19}C
$e^2/4\pi\varepsilon_0$ = 1.4400MeV fm
Fine structure constant $\alpha = \dfrac{e^2}{4\pi\varepsilon_0\hbar c}$ **= 1 / 137.036**
Boltzmann constant k = 1.3806503(24) x 10^{-23} JK^{-1} = 0.8617 x 10^{-4} eV K^{-1}

MASSES AND ENERGIES:
Atomic mass unit m_u or u = 1.66053873(13) x 10^{-27} kg
$m_u c^2$ = 931.494 MeV

Electron	m_e	= 9.10938188(72) x 10^{-31} kg
	m_e/m_u	= 5.486 x 10^{-4} = 1/1823
	$m_e c^2$	= 0.510998902(21) MeV
Proton	m_p	= 1.67262158(13) x 10^{-27} kg
	m_p / m_u	= 1.00727647
	$m_p c^2$	= 938.272 MeV

Hydrogen atom	m_H	$= 1.673533 \times 10^{-27}$ kg
	m_H / m_u	$= 1.007825$
	$m_H c^2$	$= 938.783$ MeV
Neutron	m_n	$= 1.67492716(13) \times 10^{-27}$ kg
	m_n/m_u	$= 1.00866491578(55)$
	$m_n c^2$	$= 939.565$ MeV
Alpha particle	m_α	$= 6.644656 \times 10^{-27}$ kg
	m_α / m_u	$= 4.001506175$
	$m_\alpha c^2$	$= 3727.379$ MeV

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of $\boldsymbol{L^2}$ and $\boldsymbol{L_z}$ operators with the property

$$L^2 Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\vartheta,\varphi)$$

$$L_z Y_\ell^m(\vartheta,\varphi) = m\hbar Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_{-a}^{+a} \cos^2(kx)dx = \frac{\cos(ka)\sin(ka)+ka}{k} \quad ; \quad \int_{-a}^{+a} \cos(kx)\sin(kx)dx = 0 \quad ; \quad \int_{-a}^{+a} x\cos^2(kx)dx = 0$$

$$\int_{-a}^{+a} x\cos(kx)\sin(kx)dx = \frac{2ka\cos^2(ka) - \cos(ka)\sin(ka) - ka}{2k^2} .$$

$$\int_0^\infty exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1} exp(-at^2)\, dt = \frac{n!}{2a^{n+1}}$$ with Re a > 0, n =0,1,2,..

$$\int_0^\infty t^{2n} exp(-at^2)\, dt = \frac{1.3.5......(2n-1)}{2^{n+1} a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....

Note: $$\int_{-a}^{+a} (even\ \ function\ of\ x)\, dx = 2\int_0^{+a} (even\ \ function\ of\ x)\, dx$$

$$\int_{-a}^{+a} (odd\ \ function\ of\ x)\, dx = 0 .$$