UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2008-09

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS Five PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1.

(a) A typical thermal neutron kinetic energy equals $\frac{3}{2}kT$ at T=300K. [6]

What is its velocity and its de-Broglie wavelength?

(b) Using uncertainty relation, estimate the radius of the electron whose ionization energy is 13.6 eV. [6]

(c) Explain [8]

- (i) Parity.
- (ii) Constant of motion in quantum mechanics.
- (iii) Probability interpretation of wave function.
- (iv) Complete set

(d) The wave function of a particle moving in one dimension is given by: [5]

$$\psi(x) = 0 \qquad for\ x < 0$$
$$= B\sqrt{x}\exp(-\beta x) \qquad for\ x \geq 0$$

where β is a real and positive constant.

Calculate the normalization constant B. (It is a function of β.)

Note: $\Gamma(z) = k^2 \int_0^\infty t^{z-1} \exp(-kt)\,dt \qquad \mathrm{Re}\, z > 0, \mathrm{Re}\, k > 0.$

$\Gamma(n+1) = n! \quad for\ \ n = 1,2,\ldots \ \ and \ \ \Gamma(1) = 1.$

Q.2.
Suppose that the following wave functions are a solution to the one-dimensional time independent Schrödinger equation

$$\psi_n = N_n \sin\left(\frac{nx}{a}\right), \quad n = 1,2,3,...$$

where a is some constant , N_n is the normalization and $-\infty < x < \infty$.

(i) Write an expression for the Hamiltonian and hence the Schrodinger equation which has ψ_n as its solution. What is the nature of the potential? [8]

(ii) What are the corresponding energies E_n and the full time-dependent wave functions? [4]

(iii) Are these wave functions normalizable , if so determine the coefficients N_n? [8]

(iv) Determine the expectation values of <x> and <p>. [5]

Q.3.

(a) Given that $[\, r_i\,, p_j\,] = i\hbar\ \delta_{ij}$ where $r_i = (\,x\,, y\,, z\,)$ and $p_i = (\,p_x\,, p_y\,, p_z\,)$, and $[\, L_x\,, L_y\,] = i\hbar\ L_z$ where $\vec{L} = \vec{r} \times \vec{p}$, show that

(i) $[\, x\,, p^2\,] = 2\,i\hbar\ p_x$ where $p^2 = p_x^2 + p_y^2 + p_z^2$. [5]

(ii) $[\,L_+\,, L_-\,] = 2\hbar\, L_z$ [5]

where $L_+ = L_x + i\,L_y$ and $L_- = L_x - i\,L_y$

(b) Using the relations

$$[\,\sigma_x, \sigma_y\,] = i\hbar\sigma_z \quad , \quad [\,\sigma_y, \sigma_z\,] = i\hbar\sigma_x \quad , \quad [\,\sigma_z, \sigma_x\,] = i\hbar\sigma_y$$

(i) show that $[\,\sigma_i, \sigma^2\,] = 0$ where $i = x\,, y, z$ and $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$. [10]

(ii) consider an Hamiltonian H which has following properties

$[\,\mathbf{H}\,, \boldsymbol{\sigma}^2\,] = 0$

$[\,\mathbf{H}\,, \boldsymbol{\sigma}_i\,] = 0$

Explain why the dynamical variable corresponding to only one component of the operator $\boldsymbol{\sigma}_i$ can be a measurable simultaneously with **H** and $\boldsymbol{\sigma}^2$. [5]

Q.4. Verify that the two wave functions [8]

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

and

$$\phi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

are solutions of the eigenvalue problem

$$\hat{H}\phi_n(x) = E_n\phi_n(x) \qquad \text{with} \quad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2$$

(i) Determine E_n for each of them. [5]

(ii) What is the parity of each state. [2]

(iii) Determine the solutions $\phi_0(x,y,z)$ and $\phi_1(x,y,z)$ for the Hamiltonian [10]

$$\hat{H} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] + \frac{m\omega^2}{2}\left[x^2 + y^2 + z^2\right]$$

and E_n for each of them.

Q.5. The stationary Schrödinger equation for a particle moving in a central potential $V(r)$ is

$$E\Phi(r,\vartheta,\varphi) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r}\right) + \frac{1}{2mr^2}\hat{L}^2\Phi(r,\vartheta,\varphi) + V(r)\Phi(r,\vartheta,\varphi),$$

where $\hat{L}$ is the angular momentum operator for the particle's motion.

(a) Write the wave function $\Phi(r,\theta,\phi)$ as a product of a radial [15] function $R(r)$ and an angular momentum eigenfunctions $Y_\ell^m(\vartheta,\varphi)$, and derive the differential equation for $R(r)$. State the boundary conditions on $R(r)$.

(b) An electron in the Coulomb field of a proton with Hamiltonian H is in a state described the wave function

$$\phi = \frac{1}{5}\left[4\psi_{100} + 3\psi_{211}\right]$$

(i) What is the expectation value of the energy? [5]

(ii) What is the expectation value of L^2 and L_z? [5]

Note that $\psi_{nlm} = R_{nl}(r)\, Y_l^m(\vartheta,\varphi)$ is the eigenfunction of H with energy $\frac{E_0}{n^2}$ where E_0 is a constant and

n = principal quantum number,
l = angular momentum quantum number
m = projection of angular momentum.

and $\int \psi^*_{n'l'm'}\, \psi_{nlm}\, d\tau = \delta_{n'n}\, \delta_{l'l}\, \delta_{m'm}$

@@@@END OF EXAMINATION@@@@

Appendix:

PHYSICAL CONSTANTS AND DERIVED QUANTITIES

Speed of light c = 2.99792458 x 10^8 m s^{-1} ~ 3.00 x 10^{23} fm s^{-1}
Avogadro's number N_A = 6.02214199(47) x 10^{26} molecules per kg-mole
Planck's constant h = 6.626068 76(52) x 10^{-34} J s
$\hbar$ = 1.054571 596(82) x 10^{-34} J s = 0.65821 X 10^{-21} MeV s
$\hbar^2$ = 41.802 u MeV fm^2
$\hbar c$ = 197.327 MeV fm

Fermi 1fm = 10^{-15} m

1 eV=1.6022 x 10^{-19} J 1 MeV=1.602176 x 10^{-13} J

Elementary charge e = 1.602176462(63) x 10^{-19}C , $e^2/4\pi\varepsilon_0$ = 1.4400MeV fm

Fine structure constant $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = 1/137.036$

Boltzmann constant k = 1.3806503(24) x 10^{-23} JK^{-1} = 0.8617 x 10^{-4} eV K^{-1}

MASSES AND ENERGIES

Atomic mass unit m_u or u = 1.66053873(13) x 10^{-27} kg

$m_u c^2$ = 931.494 MeV

Electron m_e = 9.10938188(72) x 10^{-31} kg = 0.510998902(21) MeV

Proton m_p= 1.67262158(13) x 10^{-27} kg = 938.272 MeV

Neutron m_n= 1.67492716(13) x 10^{-27} kg = 939.565 MeV

Hydrogen atom m_H = 1.673533 x 10^{-27} kg = 938.783 MeV

Alpha particle m_α = 6.644656 x 10^{-27} kg = 3727.379 MeV

Useful Information:

[A , CD] = [A , C] D + C [A , D]
[A C , D] = A [C, D] + [A , D] C

The functions $Y_l^m(\vartheta, \varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2 Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\vartheta,\varphi)$$

$$L_z Y_\ell^m(\vartheta,\varphi) = m\hbar Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int \sin^2(kx)\,dx = \frac{x}{2} - \frac{1}{4k}\sin(2kx)$$

$$\int_{-a}^{+a} \sin^2(kx)dx = \frac{-\cos(ka)\sin(ka) + ka}{k} \quad ; \quad \int_{-a}^{+a} \cos^2(kx)dx = \frac{\cos(ka)\sin(ka) + ka}{k}$$

$$\int_{-a}^{+a} \cos(kx)\sin(kx)dx = 0 \quad ; \quad \int_{-a}^{+a} x\cos^2(kx)dx = 0 = \int_{-a}^{+a} x\sin^2(kx)dx$$

$$\int_{-a}^{+a} x\cos(kx)\sin(kx)dx = \frac{2ka\cos^2(ka) - \cos(ka)\sin(ka) - ka}{2k^2} .$$

$$\int_0^\infty exp(-t^2)dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty t^{2n+1}\, exp(-at^2)\,dt = \frac{n!}{2a^{n+1}}$$ with Re a > 0; n =0,1,2,..

$$\int_0^\infty t^{2n}\, exp(-at^2)\,dt = \frac{1.3.5......(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

with Re a > 0 , n=0,1,2.....