UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2009/2010

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) Given $f = \rho^2 \cos(\phi) - 4z^2$,

(i) find the value of $\vec{\nabla} f$ at a point P (10 , 220^0 , - 3) , **(5 marks)**

(ii) find $\vec{\nabla} \times (\vec{\nabla} f)$ and shows that it is zero. **(5 marks)**

(b) Given $\vec{F} = \vec{e}_x (6xy) + \vec{e}_y (3x^2 - 5z^2) + \vec{e}_z (-10yz)$ and find the value of

$\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (0, 1, 0)$, $P_2 : (2, 5, 0)$ and

(i) L : a straight line from P_1 to P_2 on x – y plane , i.e., z = 0 plane , **(7 marks)**

(ii) L : a parabolic curve $y = x^2 + 1$ from P_1 to P_2 on x – y plane , i.e., z = 0 plane .

Compare this answer with that obtained in (b)(i) and comment on whether the given $\vec{F}$ is a conservative vector field or not. **(8 marks)**

Question two

Given $\vec{F} = \vec{e}_r\left(r^2\right) + \vec{e}_\theta\left(6r^2 \sin\theta\right) + \vec{e}_\phi\left(3r^2 \cos\phi\right)$,

(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if S is the closed surface enclosing the upper half spherical ball of radius 6 , i.e., $S = S_1 + S_2$ where

$$S_1 : \left(r = 6 \;,\; 0 \le \theta \le \frac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \;\;\&\;\; d\vec{s} = \vec{e}_r\, r^2 \sin\theta \, d\theta \, d\phi \xrightarrow{r=6} \vec{e}_r\, 36 \sin\theta \, d\theta \, d\phi \right)$$

$$S_2 : \left(\theta = \frac{\pi}{2} \;,\; 0 \le r \le 6 \;,\; 0 \le \phi \le 2\pi \;\;\&\;\; d\vec{s} = \vec{e}_\theta\, r \sin\theta \, dr \, d\phi \xrightarrow{\theta=\frac{\pi}{2}} \vec{e}_\theta\, r \, dr \, d\phi \right)$$

(12 marks)

(b) (i) find $\vec{\nabla} \bullet \vec{F}$, **(4 marks)**

(ii) then evaluate the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in (a) , i.e.,

$$V : \; 0 \le r \le 6 \;,\; 0 \le \theta \le \frac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \;\;\&\;\; dv = r^2 \sin\theta \, dr \, d\theta \, d\phi \,.$$

Compare this value with that obtained in (a) and make a brief comment.

(9 marks)

Question three

Given the following non-homogeneous differential equation as

$$\frac{d^2 x(t)}{dt^2} + 2\frac{d\,x(t)}{dt} + 5\,x(t) = 10\sin(t) + 17\cos(2t) \ ,$$

(a) find its particular solution $x_p(t)$, **(8 marks)**

(b) for the homogeneous part of the given non-homogeneous differential equation , i.e., $\frac{d^2 x(t)}{dt^2} + 2\frac{d\,x(t)}{dt} + 5\,x(t) = 0$, set $x(t) = \sum_{n=0}^{\infty} a_n\, t^{n+s}$ & $a_0 \neq 0$ and utilize the power series method to find its two independent solutions in power series form truncated up to a_4 term. **(15 marks)**

(c) write down the general solution of the given non-homogeneous differential equation in terms of the answers obtained in (a) & (b) . **(2 marks)**

Question four

(a) Given the following partial differential equation as

$$\frac{x^2\, y}{3}\frac{\partial f(x,y)}{\partial x}+\frac{2}{x^2\, y^2}\frac{\partial f(x,y)}{\partial y}=0$$

, use separation variable scheme to deduce two ordinary differential equations. **(5 marks)**

(b) An elastic string of length 5 is fixed at its two ends, i.e., at $x=0$ & $x=5$ and its transverse deflection $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=4\frac{\partial^2 u(x,t)}{\partial x^2},$$

(i) by direct substitution, show that . $u(x,t)=\sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{5}\right)\cos\left(\frac{2n\pi t}{5}\right)$

where E_n $n=1,2,3,\cdots\cdots$ are arbitrary constants , satisfies two fixed end conditions , i.e., $u(0,t)=0=u(5,t)$, as well as zero initial speed condition ,

i.e., $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=0$. **(8 marks)**

(ii) then find E_n in terms of n if the initial position of the string , i.e., $u(x,0)$, is given as $u(x,0)=\begin{cases} 2x & if \quad 0\le x\le 3 \\ -3x+15 & if \quad 3\le x\le 5 \end{cases}$

(hint : $\int_{x=0}^{5} \sin\left(\frac{n\pi x}{5}\right)\sin\left(\frac{m\pi x}{5}\right)dx=\begin{cases} 0 & if \quad n\ne m \\ \frac{5}{2} & if \quad n=m \end{cases}$ &

$\int x \sin\left(\frac{n\pi x}{5}\right)dx=\frac{25}{n^2\pi^2}\sin\left(\frac{n\pi x}{5}\right)-\frac{5}{n\pi}x\cos\left(\frac{n\pi x}{5}\right)$) **(12 marks)**

Question five

Given the following coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -20\, x_1(t) + 8\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 5\, x_1(t) - 26\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\,X = -\omega^2\, X$ where

$$A = \begin{pmatrix} -5 & 16 \\ 4 & -17 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **(4 marks)**

(b) find the eigenfrequencies ω of the given coupled system, **(5 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b), **(6 marks)**

(d) write down the general solutions for $x_1(t)$ & $x_2(t)$. If the initial conditions are given as

$$x_1(0) = -3\ ,\ x_2(0) = +2\ ,\ \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} = 4\ \&\ \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} = 0$$, find the specific

solutions for $x_1(t)$ & $x_2(t)$. **(10 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system