UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2009/2010

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF SIX QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

Question one

(a) (i) Given P(10 , 300^0 , - 7) in cylindrica coordinate system, find its Cartesian and spherical coordinates . **(5 marks)**

(ii) Given P(8 , 120^0 , 200^0) in spherical coordinate system, find its cylindrical and Cartesian coordinates . **(5 marks)**

(b) (i) Draw the cylindrical unit vectors $\vec{e}_\rho$ & $\vec{e}_\phi$ as well as the cartesian unit vectors $\vec{e}_x$ & $\vec{e}_y$ on $z = 0$ *plane*, i.e., x – y plane. **(5 marks)**

(ii) express $\vec{e}_\rho$ & $\vec{e}_\phi$ in terms of $\vec{e}_x$ & $\vec{e}_y$ and deduce that

$$\frac{\partial \vec{e}_\phi}{\partial \phi} = - \vec{e}_\rho$$ **(5 marks)**

(c) Given a scalar function f in Cartesian system as $f = 8\,x^4 - 2\,y\,z^3$,

find $\vec{\nabla} f$ at the point P (x = - 3 , y = 4 , z = - 2) . **(5 marks)**

Question two

Given a vector field $\vec{F} = \vec{e}_\rho \, 6\,\rho^2 + \vec{e}_\phi \, \rho^2 + \vec{e}_z \, z^2 \sin(\phi)$

(a) evaluate the value of the closed loop line integral $\oint_L \vec{F} \bullet d\vec{l}$ if L is a circular closed loop of radius 7 in counter clockwise sense on $z = 0$ *plane* and centred at the origin, i.e., $z = 0$, $\rho = 7$, $0 \le \phi \le 2\pi$ & $d\vec{l} = \vec{e}_\phi \, \rho \, d\phi \xrightarrow{\rho = 7} \vec{e}_\phi \, 7 \, d\phi$

(9 marks)

(b) (i) find $\vec{\nabla} \times \vec{F}$, **(6 marks)**

(ii) find the value of the surface integral $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is the surface enclosed by the given closed loop in (a) , i.e.,

$d\vec{s} = \vec{e}_z \, \rho \, d\rho \, d\phi$, $z = 0$, $0 \le \rho \le 7$ & $0 \le \phi \le 2\pi$

Compare the result here with that obtained in (a) and make brief comment on Stokes' theorem. **(10 marks)**

Question three

Given the following differential equation as :

$$\left(1 - x^2\right)\frac{d^2\, y(x)}{d\,x^2} - 2\,x\,\frac{d\,y(x)}{d\,x} + 6\,y(x) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$,

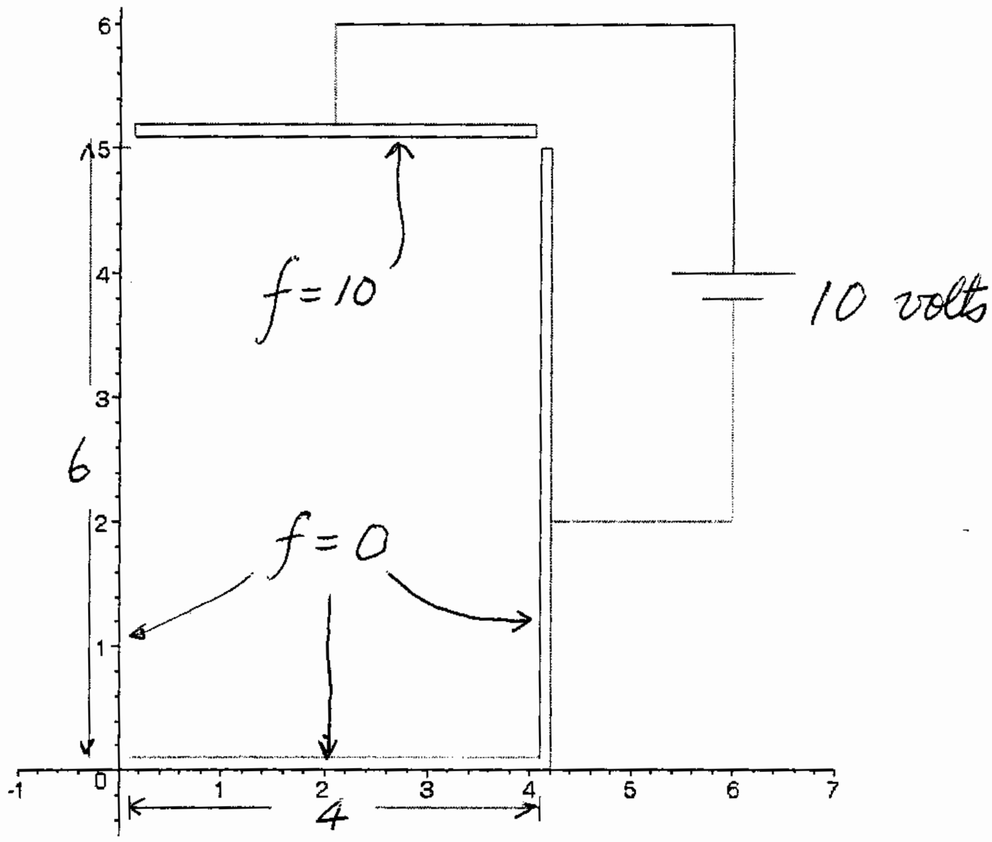
(a) write down the indicial equations. Find the values of s and a_1 . **(10 marks)**

(b) write down the recurrence relation. For all the appropriate values of s and a_1 found in (a), set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_5 . Thus write down two independent solution in their polynomial forms.

(15 marks)

Question four

An U – tube capacitor extended very long into z direction with its x – y cross section as shown below :



Its electric potential $f(x,y)$ satisfies the following two dimensional Laplace equation :

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$$

(a) set $f(x,y) = F(x)\,G(y)$ and use separation scheme to deduce the following ordinary differential equations :

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = -k^2 F(x) \\ \dfrac{d^2 G(y)}{d y^2} = +k^2 G(y) \end{cases} \quad \text{where } k \text{ is a constant}$$

(6 marks)

(b) write the general solution for (a) as

$$f(x,y) = \sum_{\forall k} f_k(x,y)$$
$$= \sum_{\forall k} \left(A_k \cos(k\,x) + B_k \sin(k\,x)\right)\left(C_k \cosh(k\,y) + D_k \sinh(k\,y)\right)$$

where A_k , B_k , C_k & D_k are arbitrary constants.

(i) Apply three zero boundary conditions , i.e.,

$f_k(0,y) = 0$, $f_k(4,y) = 0$ & $f_k(x,0) = 0$,

and show that the general solution can be simplified as :

$$f(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{4}\right)\sinh\left(\frac{n\pi y}{4}\right)$$ **(10 marks)**

(ii) Apply the non-zero boundary condition , i.e., $f(x,6) = 10 \quad \forall\ x$, and use the

orthogonal condition $\int_{x=1}^{4} \sin\left(\frac{n\,\pi\,x}{4}\right)\sin\left(\frac{m\,\pi\,x}{4}\right)d\,x = \begin{cases} 0 & if \quad m \neq n \\ 2 & if \quad m = n \end{cases}$

to deduce that $E_n = \begin{cases} \dfrac{40}{n\,\pi\sinh\left(\dfrac{3\,n\,\pi}{2}\right)} & if \quad n = 1,3,5,\cdots\cdots \\ 0 & if \quad n = 2,3,6,\cdots\cdots \end{cases}$

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(9 marks)

Question five

(a) Given a simple harmonic oscillator governed by the following equation

$2\,\dfrac{d^2 x(t)}{d x^2} = -10\,x(t)$, find the values of the angular frequency, frequency and period of the harmonic solution of $x(t)$. **(6 marks)**

(b) Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{d t^2} = -7\,x_1(t) + 3\,x_2(t) \\ \dfrac{d^2 x_2(t)}{d t^2} = 4\,x_1(t) - 8\,x_2(t) \end{cases}$$

(i) set $x_1(t) = X_1\,e^{i\omega t}$ & $x_2(t) = X_2\,e^{i\omega t}$, deduce the following matrix equation $A\,X = -\omega^2\,X$ where

$A = \begin{pmatrix} -7 & 3 \\ 4 & -8 \end{pmatrix}$ & $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ **(5 marks)**

(ii) find the eigenfrequencies ω of the given coupled system , **(7 marks)**

(iii) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b). **(7 marks)**

Question six

Given the following non-homogeneous differential equation as :

$$\frac{d^2 x(t)}{dt^2} + 2\frac{d x(t)}{dt} + 5\,x(t) = 6\,t - 5\,t^2$$

(a) set its particular solution as $x_p(t) = k_1\,t^2 + k_2\,t + k_3$, determine the values of k_1 , k_2 & k_3 . **(9 marks)**

(b) find the general solution to the homogeneous part of the given differential equation , i.e.,

$$\frac{d^2 x(t)}{dt^2} + 2\frac{d x(t)}{dt} + 5\,x(t) = 0$$, and name it as $x_h(t)$. **(6 marks)**

(c) write the general solution to the given non-homogeneous differential equation as $x(t) = x_p(t) + x_h(t)$ and determine the values of the arbitrary constants in this general solution if the initial conditions are given as $x(0) = -3$ & $\left.\frac{d x(t)}{dt}\right|_{t=0} = 2$.

(10 marks)

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system