UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2009_10

TITLE OF THE PAPER: **CLASSICAL MECHANICS**

COURSE NUMBER : **P320**

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

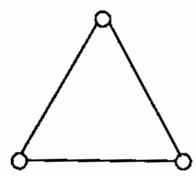
THIS PAPER HAS **SIX** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1:

(A) (i) Define generalized coordinates and canonical coordinates. Explain the difference between them. [2]

(ii) Explain the term " the degrees of freedom in a dynamical system". For the following planar system of three atoms of equal mass m with the conformation of equilateral triangle, state the number of degrees of freedom the system has with reference to a space fixed co-ordinate system. [5]



Assume the distance between the atoms along the bond is α .

(iii) What do you understand by the term 'cyclic coordinate' ? What is the implication of existence of such a coordinate in a dynamical system? [2]

(iv) Explain the term " conservative system" . In such a system, what is the general relation between force and the potential. [2]

(v) Explain the term " rigid body ". How many coordinates are necessary to describe the motion of a rigid body? [2]

(B) Consider a particle of mass m confined to a motion on a plane with external force given by the potential V(r) where r is the radial distance w.r.t. space fixed co-ordinate system. Using plane polar co-ordinates,

(i) Define the Lagrangian for the system. [4]

(ii) Derive the equations of motion using Lagrange's equations. [8]

Q.2.

(A) Consider a two particle system with masses m_1 and m_2. With reference to a space fixed co-ordinate system, the two particles have following co-ordinates:

Particle 1: $\vec{r}_1$ and Particle 2: $\vec{r}_2$.

(i) Define the centre of mass co-ordinate $\vec{R}$ for this system. [2]

(ii) Show that the kinetic energy T for the system is given by [5]

$$T = \frac{1}{2} M V_c^2 + \mu v^2$$

where $\vec{V}_c = \dot{\vec{R}} = velocity\ of\ centre\ of\ mass$. $\vec{v} = \vec{v}_1 - \vec{v}_2 = relative\ velocity$.

$\vec{v}_1 = velocity\ of\ particle\ 1$. $\vec{v}_2 = velocity\ of\ particle\ 2$.

$M = m_1 + m_2$ and $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.

(B) If the origin of the center of mass system coincides with the Sun in a planetary motion, show that this implies that the Sun has infinite mass. [3]

(C) For a two body problem with the centre of mass as the origin, following relation is given:

$$\dot{r} = \left[\frac{2}{\mu}\left(E - V(r) - \frac{\ell^2}{2\mu r^2}\right)\right]^{\frac{1}{2}}$$

where E =Total energy, ℓ = Orbital angular momentum and μ =Reduced mass.

Assume that the potential can be of the form $V(r) = -\frac{k}{r}$ which is attractive or $V(r) = \frac{k}{r}$ which is repulsive for k >0 .

(i) Explain why only for attractive potential it is possible to have bound states provided E<0. [5]

(ii) Describe the kind orbits one can have for bound states. [5]

(iii) What is the value of the radial velocity at turning points of a bound orbit? [1]

(iv) For a body moving under the influence of potential

$$V(r) = -\frac{k}{r}$$

The orbit equation is given by

$$\frac{1}{r} = \frac{\mu k}{l^2}[1 + \varepsilon \cos\theta]$$

Here ε is the eccentricity of the orbit.

For a circular orbit of radius r_0 and velocity v_0 , show that $\frac{k}{\mu r_0} = v_0^2$ [4]

Q.3.

(A) Consider free small oscillations in one dimension about a position of stable equilibrium of a particle with mass m and displacement as x .

(i) Show that the potential is given by $\frac{1}{2} k x^2$. [5]

(ii) Derive an expression for the Lagrangian. [1]

(iii) Derive the equations of motion for this system. [3]

(iv) Show that angular frequency depends only on the property of the mechanical system. [2]

(v) Show that the general solution of the equation of motion is [4]

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) = a \cos(\omega t + \alpha)$$

$$where \ \tan(\alpha) = -\frac{c_2}{c_1} \quad and \quad a = \sqrt{c_1^2 + c_2^2}$$

Here $\omega = \sqrt{\frac{k}{m}}$ and c_1, c_2 = constants.

(B) In the case of two coupled oscillators of equal mass m the equations of motion for normal modes are given as

$$\ddot{\bar{x}}_1 + \frac{k}{m}\bar{x}_1 = 0$$

$$\ddot{\bar{x}}_2 + \frac{k''}{m}\bar{x}_2 = 0 \qquad where \quad k'' = k + 2k'$$

Here $\bar{x}_1 = x_1 + x_2$ and $\bar{x}_2 = x_1 - x_2$ where x_1 and x_2 as displacement of particles from their respective equilibrium positions .

(i) What are the frequencies of the normal modes. [5]

(ii) Illustrate the normal vibrations with diagrams. [5]

Q.4.

(A) For any vector $\vec{A}$,

$$\left(\frac{d\,\vec{A}}{d\,t}\right)_{fixed} = \left(\frac{d\,\vec{A}}{d\,t}\right)_{rotating} + \vec{\omega} \times \vec{A}$$

where $\vec{\omega}$ = uniform angular velocity.
Velocity $\vec{v}$ relative to a inertial system whose origin coincides with center of mass system is given as

$$\vec{v} = \vec{v}_r + \vec{\omega} \times \vec{r}$$

$\vec{v}_r$ = velocity of the particle in the rotational co-ordinate system

and $\vec{r}$ = radial vector .

(i) Show that to an observer in a rotating system the effective force on a particle of mass m is given by [5]

$$\vec{F}_{eff} = \vec{F} - m\,\vec{\omega} \times (\,\vec{\omega} \times \vec{r}\,) - 2m\,\vec{\omega} \times \vec{v}_r$$

$\vec{F}$ is the force in the inertial coordinate system.

(ii) What is the interpretation of the term $m\vec{\omega} \times \vec{\omega} \times \vec{r}$. [3]

(iii) Explain the physical significance of the last term $m\vec{\omega} \times \vec{v}_r$ in the relation for effective force. [5]

(B) The rotational motion of a rigid body, relative to body axes , Euler's equations of motion are

$$\frac{dL_x}{dt} + \omega_y L_z - \omega_z L_y = N_x$$

$$\frac{dL_y}{dt} + \omega_z L_x - \omega_x L_z = N_y$$

$$\frac{dL_z}{dt} + \omega_x L_y - \omega_y L_x = N_z$$

where, for i = x, y, z :

L_i = total angular momentum components,

ω_i = angular velocity components.

N_i = moment of force or torque components.

Assuming there are no external forces and body axes are principal axes, then $L_i = I_i \omega_i$ where I_i are principal moments of inertia.

For symmetrical body and taking symmetry axis as the z-principal axis so that $I_x = I_y$,

(i) Show that z-component of angular velocity is constant, [2]

(ii) Show that kinetic energy $T = \frac{1}{2} I_x A^2 + \frac{1}{2} I_z \omega_z^2$ [5]

(iii) Total angular momentum $L^2 = I_x^2 A^2 + I_z^2 \omega_z^2$. [5]

where $A^2 = \omega_x^2 + \omega_y^2$.

Q.5.

(A) With $H = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t)$ where L is a Lagrangian, show that

(i) $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$ [5]

(ii) Show that for a function $u(q_i, p_i, t)$, [5]

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

Use the identity to show that , if u does not depend on time explicitly and if u is a constant of motion then

$$[u, H] = 0$$

(B) Consider for one dimensional motion of a particle of mass *m* under the influence of potential of the form

$$V(x) = kx$$

where k is force constant.

(i) Write down the Hamiltonian in terms canonical variables. [5]

(ii) For an equation of the type

$$\frac{du}{dt} = [u, H]$$

the formal solution is given by the series expansion

$$u(t) = u_0 + t[u, H]_0 + \frac{t^2}{2!}[[u, H], H]_0 + \frac{t^3}{3!}[[[u, H], H], H]_0 + ...$$

where subscript *0* denotes the initial conditions at *t=0.*

Use the above relation to show that for the given Hamiltonian, the complete solution is given by a series [10]

$$x = x_0 + \frac{p_0 t}{m} - \frac{kt^2}{2m}$$

where x_0 and p_0 are position and momentum at t=0.

@@@@END OF EXAMINATION@@@@

Appendix:

(i) Plane polar coordinates are defined by the relation:

$$x = r\cos(\vartheta) \quad and \quad y = r\sin(\vartheta)$$

(ii) Spherical polar coordinates are defined by

$$x = r\sin(\vartheta)\cos(\varphi)$$

$$y = r\sin(\vartheta)\sin(\varphi)$$

$$z = r\cos(\vartheta)$$

(iii) Lagrange's equations are given by: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$

(iv) Hamilton's equations are given by the relations:

$$\frac{\partial q_i}{\partial t} = [q_i, H] \quad and \quad \frac{\partial p_i}{\partial t} = [p_i, H]$$

where H is the Hamiltonian of the dynamical system.

(v) Poisson bracket of two functions with respect to canonical variables *(q,p)* is defined by

$$[u,v]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} \right)$$

Following Poisson brackets have the properties:

$$[q_i, q_j] = 0 = [p_i, p_j]$$

$$[q_i, p_j] = \delta_{ij} = -[p_i, q_j]$$

(vi) Following properties of Poisson brackets are useful:

$$[u,u] = 0 \quad , \quad [u,v] = -[v,u]$$

$$[au + bv, w] = a[u,w] + b[v,w] \quad \textit{where a and b are constants.}$$

$$[uv, w] = u[v,w] + [u,w]v$$

$$[u, vw] = v[u,w] + [u,v]w$$

Jacobi Identity: $[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$