UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2009/2010

TITLE OF PAPER : ELECTROMAGNETIC THEORY

COURSE NUMBER : P331

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS NINE PAGES, INCLUDING THIS PAGE.

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P331 ELECTROMAGNETIC THEORY

Question one

(a) (i) From the conservation of electric charges, i.e.,

$$I = -\frac{\partial q}{\partial t} \quad where \quad I = \oint_S \vec{J} \bullet d\vec{s} \quad \& \quad q = \oint_V \rho_v \, dv \, ,$$

deduce the following equation of continuity for electric charges

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho_v}{\partial t}$$ **(4 marks)**

(ii) Show that without introducing the displacement current term, i.e., $\frac{\partial \vec{D}}{\partial t}$, in the equation for Ampere's law, i.e., $\vec{\nabla} \times \vec{H} = \vec{J}$ instead of $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, Maxwell's equations would contradict with the continuity equation for electric charges. **(7 marks)**

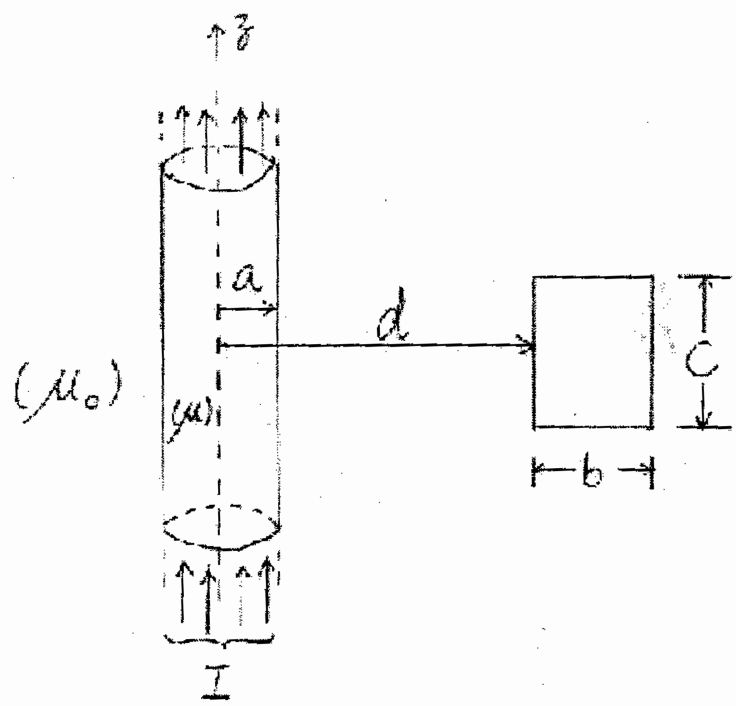
(b) A spherical ball of radius R_0 with a dielectric constant ε , centered at the origin , carries a charge distribution of $\rho_v = 12\left(1 + \alpha\, r\right)$ and embeds in air of ε_0

(i) use integral Gauss law , choose proper Gaussian surfaces to find $\vec{E}$ in terms of r , R_0 & α for $0 \le r \le R_0$ & $r \ge R_0$ regions , **(12 marks)**

(ii) determine the value of α such that the electric field everywhere outside the spherical ball is zero. **(2 marks)**

Question two

(a) A very long straight wire of cross section radius a with a permeability μ, with its central axis coincide with the z – axis, carries an uniform current density in the positive z direction with the total static current of I, i.e., the current density inside the wire is $\frac{I}{\pi a^2}$, as shown in the diagram below :



Use integral Ampere's law, choose proper closed loops to find $\vec{B}$ in terms of ρ, a & I for $0 \le \rho \le a$ & $\rho \ge a$ regions. **(12 marks)**

(b) Placing a rectangular conducting loop of dimension $b \times c$ a distance of $d\,(> a)$ away from the central axis of the wire as shown in the diagram in (a), i.e., the inner region confined by the rectangular loop in clockwise sense is

$S:\ d \le \rho \le d+b \ ,\ 0 \le z \le c \ \&\ d\vec{s} = \vec{a}_\phi \, d\rho \, dz$,

(i) find the total magnetic flux passing through the inner region confined by the rectangular loop i.e., $\int_S \vec{B} \bullet d\vec{s}$, in terms of a, b, c, d & I. **(10 marks)**

(ii) If the wire carries a sinusoidal current of $I_0 \sin(\omega t)$ instead of carrying a static current I, find the induced e.m.f. in the rectangular conducting loop in terms of a, b, c, d, ω & I_0 under quasi static situation. **(3 marks)**

Question three

(a) In a conductive region, based on Drude's model the force on a conduction electron by $\vec{E}$ is $-e\vec{E}$ and the retardation force by the ion lattice of the conductor is $-\dfrac{m_e \vec{v}_d}{\tau_c}$ and thus the equation of motion for an average conduction electron in the conductor is

$$m_e \frac{d\vec{v}_d}{dt} = -e\vec{E} - \frac{m_e \vec{v}_d}{\tau_c} ,$$

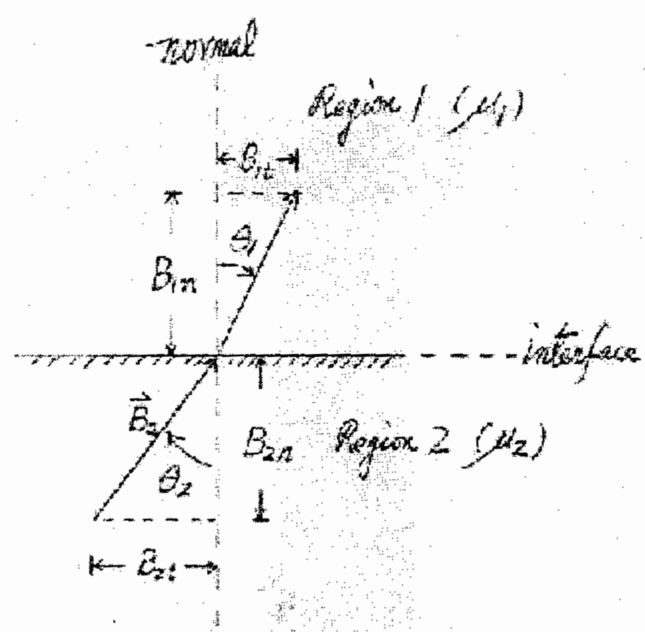
(i) explain briefly $\vec{v}_d$ & τ_c , **(2 marks)**

(ii) in the steady state situation , i.e., $\dfrac{d\vec{v}_d}{dt} = 0$, deduce the following point form of Ohm's law $\vec{J} = \sigma \vec{E}$ *where* $\sigma = \dfrac{n e^2}{m_e}\tau_c$, (Hint : $\vec{J} = \rho_v \vec{v}_d = -n e \vec{v}_d$)

(6 marks)

(iii) if a certain pure metal having an atomic density of $5.8 \times 10^{28} \dfrac{atom}{m^3}$ at room temperature and two outer orbit electrons available for conduction, find the value of τ_c if its measured dc conductivity is $\sigma = 3.6 \times 10^7 \dfrac{1}{m\,\Omega}$. **(5 marks)**

(b) Given an interface separating two isotropic materials of permeabilities μ_1 & μ_2 , $\vec{B}_1$ & $\vec{B}_2$ are the magnetic field at points just to either side of the interface infinitely close to each other and θ_1 & θ_2 are their respective angles made with the normal as shown in the diagram below



Question three (continued)

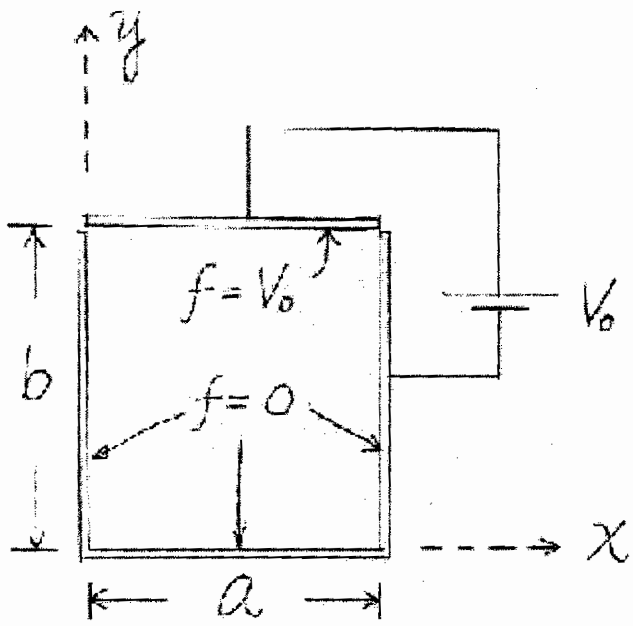
(i) use integral magnetic Gauss law , choosing proper Gaussian surface (pillbox in shape across the interface) to deduce that the normal component of $\vec{B}$ is continuous at the interface , i.e., $B_{1n} = B_{2n}$, **(6 marks)**

(ii) together with the tangential component of $\vec{H}$ is continuous at the interface , i.e., $H_{1t} = H_{2t}$, deduce the following refraction relation for $\vec{B}$ as

$$\tan(\theta_2) = \frac{\mu_2}{\mu_1}\tan(\theta_1)$$ **(6 marks)**

Question four

An infinitely long , rectangular U shaped conducting channel is insulated at the corners from the conducting plate forming the fourth side with interior dimensions as shown below :



The electric potential in between the two conductors $f(x,y)$ satisfies 2 – D Laplace equation ,

i.e., $\dfrac{\partial^2 f(x,y)}{\partial x^2} + \dfrac{\partial^2 f(x,y)}{\partial y^2} = 0$

(a) Setting $f(x,y) = X(x)\,Y(y)$ and using separation of variable scheme to break 2 – D Laplace equation into two ordinary differential equations , **(5 marks)**

(b) by direct substitution, show that $f_n(x,y) = E_n \sin\left(\dfrac{n\pi x}{a}\right)\sinh\left(\dfrac{n\pi y}{a}\right)$, where E_n *with* $n = 1,2,3,\cdots$ are constants , not only satisfies 2 – D Laplace equation but also satisfies three zero boundary conditions , i.e., $f_n = 0$ at $x = 0$, $x = a$ & $y = 0$, **(8 marks)**

(c) apply the final non-zero boundary condition , i.e., $f(x,b) = V_0$, to the following $f(x,y) = \sum_{n=1}^{\infty} f_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\dfrac{n\pi x}{a}\right)\sinh\left(\dfrac{n\pi y}{a}\right)$ to find E_n in terms of V_0 , a , b & n .

(Hint : $\int_0^a \sin\left(\dfrac{n\pi x}{a}\right)\sin\left(\dfrac{m\pi x}{a}\right)dx = \begin{cases} 0 & if \quad n \neq m \\ \dfrac{a}{2} & if \quad n = m \end{cases}$) **(12 marks)**

Question five

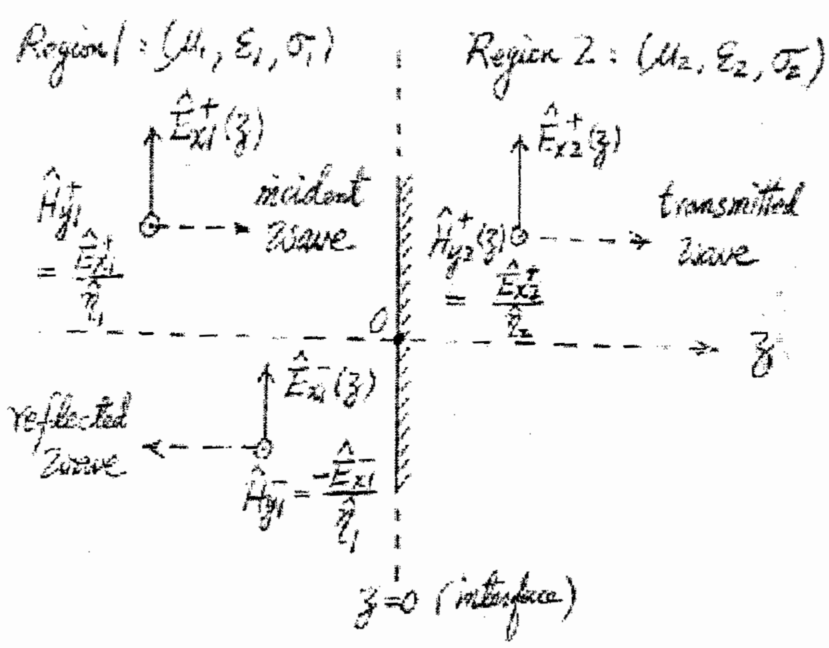
(a) An uniform plane wave traveling along + z direction with the field components $E_x(z)$ & $H_y(z)$ has a complex electric field amplitude $\hat{E}_m = 50\, e^{i\, 40^0}\ \frac{V}{m}$ and propagates at a frequency $f = 6 \times 10^6\ Hz$ in a material region having the parameters of $\mu = \mu_0$, $\varepsilon = 4\, \varepsilon_0$ & $\frac{\sigma}{\omega\, \varepsilon} = 1$,

(i) find the values of the propagation constant $\hat{\gamma}\ (= \alpha + i\, \beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave , **(4 marks)**

(ii) express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (a)(i) inserted , **(5 marks)**

(iii) find the values of the penetration depth , wave length and phase velocity of the given wave . **(3 marks)**

(b) An uniform plane wave is normally incident upon an interface separating two regions . The incident wave is given as $\left(\hat{E}^+_{x1} = \hat{E}^+_{m1}\, e^{-\hat{\gamma}_1 z} ,\ \hat{H}^+_{y1} = \frac{\hat{E}^+_{m1}}{\hat{\eta}_1}\, e^{-\hat{\gamma}_1 z} \right)$ and thus the reflected and transmitted wave can be written as $\left(\hat{E}^-_{x1} = \hat{E}^-_{m1}\, e^{+\hat{\gamma}_1 z} ,\ \hat{H}^-_{y1} = -\frac{\hat{E}^-_{m1}}{\hat{\eta}_1}\, e^{+\hat{\gamma}_1 z} \right)$ and $\left(\hat{E}^+_{x2} = \hat{E}^+_{m2}\, e^{-\hat{\gamma}_2 z} ,\ \hat{H}^+_{y2} = \frac{\hat{E}^+_{m2}}{\hat{\eta}_2}\, e^{-\hat{\gamma}_2 z} \right)$ respectively as shown below



Question five (continued)

(i) from the boundary conditions at the interface , i.e., both total $\hat{E}_x$ & $\hat{H}_y$ are continuous at $z = 0$, deduce the following

$$\begin{cases} \hat{E}^-_{m1} = \hat{E}^+_{m1} \dfrac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \\ \hat{E}^+_{m2} = \hat{E}^+_{m1} \dfrac{2\,\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_1} \end{cases}$$

(9 marks)

(ii) if region1 is air (i.e., $\mu_1 = \mu_0$, $\varepsilon_1 = \varepsilon_0$, $\sigma_1 = 0$) , region 2 is a lossless medium with parameters of $\mu_2 = \mu_0$, $\varepsilon_2 = 9\,\varepsilon_0$, $\sigma_2 = 0$, and the incident complex amplitude is $\hat{E}^+_{m1} = 80\, e^{i\,50^0} \dfrac{V}{m}$, find the values of $\hat{E}^-_{m1}$ & $\hat{E}^+_{m2}$.

(4 marks)

Useful informations

$$e = 1.6\times10^{-19}\ C \quad , \quad m_e = 9.1\times10^{-31}\ kg \quad , \quad \mu_0 = 4\,\pi\times10^{-7}\ \frac{H}{m} \quad , \quad \varepsilon_0 = 8.85\times10^{-12}\ \frac{F}{m}$$

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1} \quad , \quad \beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1} \quad , \quad \frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3\times10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)} \quad , \quad \eta_0 = 120\,\pi\ \Omega = 377\ \Omega \quad , \quad \beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv \ , \ \oiint_S \vec{B}\bullet d\vec{s} \equiv 0 \ , \ \oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho_v}{\varepsilon} \quad , \quad \vec{\nabla}\bullet\vec{B} = 0 \quad , \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad , \quad \vec{\nabla}\times\vec{B} = \mu\,\vec{J} + \mu\,\varepsilon\frac{\partial\vec{E}}{\partial t} \quad , \quad \vec{J} = \sigma\,\vec{E}$$

$$\oiint_S \vec{F}\bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla}\bullet\vec{F}\right)dv \quad , \quad \oint_L \vec{F}\bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla}\times\vec{F}\right)\bullet d\vec{s}$$

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{F}\right) \equiv 0 \quad , \quad \vec{\nabla}\times\left(\vec{\nabla}\,f\right) \equiv 0 \quad , \quad \vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla}\bullet\vec{F}\right) - \nabla^2\,\vec{F}$$

$$\vec{\nabla}\,f = \vec{e}_1\,\frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2\,\frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3\,\frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla}\bullet\vec{F} = \frac{1}{h_1\,h_2\,h_3}\left(\frac{\partial(F_1\,h_2\,h_3)}{\partial u_1} + \frac{\partial(F_2\,h_1\,h_3)}{\partial u_2} + \frac{\partial(F_3\,h_1\,h_2)}{\partial u_3}\right)$$

$$\vec{\nabla}\times\vec{F} = \frac{\vec{e}_1}{h_2\,h_3}\left(\frac{\partial(F_3\,h_3)}{\partial u_2} - \frac{\partial(F_2\,h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1\,h_3}\left(\frac{\partial(F_1\,h_1)}{\partial u_3} - \frac{\partial(F_3\,h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1\,h_2}\left(\frac{\partial(F_2\,h_2)}{\partial u_1} - \frac{\partial(F_1\,h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1\,F_1 + \vec{e}_2\,F_2 + \vec{e}_3\,F_3$ and

$(u_1\,,u_2\,,u_3)$	represents	$(x\,,y\,,z)$	for rectangular coordinate system
	represents	$(\rho\,,\phi\,,z)$	for cylindrical coordinate system
	represents	$(r\,,\theta\,,\phi)$	for spherical coordinate system
$(\vec{e}_1\,,\vec{e}_2\,,\vec{e}_3)$	represents	$(\vec{e}_x\,,\vec{e}_y\,,\vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho\,,\vec{e}_\phi\,,\vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r\,,\vec{e}_\theta\,,\vec{e}_\phi)$	for spherical coordinate system
$(h_1\,,h_2\,,h_3)$	represents	$(1\,,1\,,1)$	for rectangular coordinate system
	represents	$(1\,,\rho\,,1)$	for cylindrical coordinate system
	represents	$(1\,,r\,,r\sin(\theta))$	for spherical coordinate system