UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2009-10

TITLE OF THE PAPER: QUANTUM MECHANICS-I

COURSE NUMBER : P342

TIME ALLOWED : THREE HOURS

INSTRUCTIONS:

- ANSWER ANY **FOUR** OUT OF **FIVE** QUESTIONS.
- EACH QUESTION CARRIES **25** MARKS.
- MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.
- USE THE INFORMATION GIVEN IN THE ATTACHED **APPENDIX** WHEN NECESSARY.

THIS PAPER HAS **Six** PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL THE INVIGILATOR HAS GIVEN PERMISSION.

Q.1. (a) What is the de-Broglie wavelength of neutron with a kinetic energy of 1.7 MeV? Will such neutrons produce a diffraction pattern in a crystal of lattice point distances of the order of 10^{-8} m? [3]

(b) Using the uncertainty relation, show that in a nucleus with average potential energy <U> ≥ 15 Mev, the bound nucleon is confined within a sphere of radius $r_0 \geq 1.2 \times 10^{-15}$ m. [2]

(c) Explain [6]

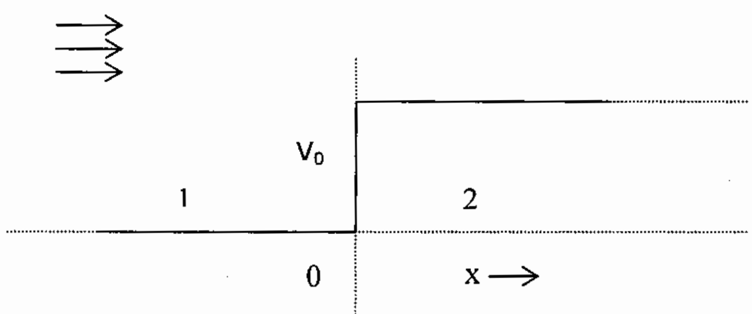
(i) Superposition principle and its importance in quantum mechanics.

(ii) Why it is necessary to have only linear operators associated with a dynamical variable.

(iii) Probability interpretation of wave function and its implications.

(d) Consider the step potential

$$V(x) = V_0 \quad x > 0$$
$$\quad\;\; = 0 \quad x<0$$



Consider a current of particles of mass m propagating from left to right of energy $E > V_0$.

Define $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$, $k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$

The general solutions for the regions *1 (x<0)* and *2(x >0)* are

$$\phi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} \quad , \quad \phi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$$

(i) State the boundary conditions on the solutions. [2]

(ii) Show that $\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$ *and* $\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$. [10]

(iii) What is the interpretation of $\frac{B_1}{A_1}$ and $\frac{A_2}{A_1}$? [2]

Q.2. (a) Explain the following:

(i) What is the difference between a state of the system given by the ket $|\ell\, m\rangle$ and the wave function $\varphi_{\ell m}(r,\vartheta,\varphi)$ describing the same state. [2]

(ii) A dynamical quantity is always represented by a Hermitian linear operator. [2]

(b) (i)A Hamiltonian H has two eigenfunctions ψ_0 and ψ_1 belonging to two different energies E_0 and E_1 respectively. Show that the two eigenfunctions are orthogonal . [5]

(ii) If the two eigenfunctions belong to the same energy, show that they need not be orthogonal. [3]

(c) Consider the one dimensional problem of a particle of mass m in a potential

$$V(x) = \begin{cases} \infty, & x \le 0, \\ 0 & 0 \le x \le a, \\ V_0 & x \ge a. \end{cases}$$

(i) Sketch the potential, [1]
(ii) Find the solutions in the three regions for $E < V_0$, [6]
(iii) Show that the bound state energies ($E < V_0$) are given by the equation [6]

$$\tan\left(\frac{\sqrt{2mE}\,a}{\hbar}\right) = -\sqrt{\frac{E}{V_0 - E}}$$

Q.3. (a) Verify that the two wave-functions [4]

$$\phi_0(x) = \left(\frac{a}{\pi}\right)^{1/4} \exp(-ax^2/2)$$

$$\phi_1(x) = \left(\frac{a}{\pi}\right)^{1/4} (2a)^{1/2} \quad x\exp(-ax^2/2)$$

are eigenfunctions of the Hamiltonian

$$H = \frac{\hbar^2}{2m}\left(-\frac{d^2}{dx^2} + a^2x^2\right)$$

belonging to two different energies.

(i) Show that the two eigenfunctions are normalized. [3]
(ii) Show that $\phi_0(x)$ and $\phi_1(x)$ are orthogonal. [2]
(iii) What are the parities of the two wave functions. [2]

(iv) Determine the value of $\Delta x = \sqrt{< x^2 > - < x >^2}$ corresponding to $\phi_0(x)$. [4]
Here < > corresponds to expectation value.

(b) Determine the energies for the ground state $\phi_0(x,y)$ and the first excited state $\phi_1(x,y)$ for the Hamiltonian [10]

$$H = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{m\omega^2}{2}\left[x^2 + y^2\right]$$

Use the result

$$-\frac{\hbar^2}{2m}\frac{d^2 u_n(x)}{dx^2} + \frac{m\omega^2}{2}x^2 u_n(x) = (n + \tfrac{1}{2})\hbar\omega u_n(x) \text{ where } n=0,1,2,\ldots$$

Comment on the degeneracy of the states.

Q.4. (a) Using the relation $[x_i, p_j] = i\hbar\delta_{ij}$, where $i, j = x,y,z$ and $\bar{L} = \bar{r} \times \bar{p}$
show that

(i) $[L_z, x] = i\hbar y$; $[L_z, y] = -i\hbar x$ and $[L_z, z] = 0$ [6]
(ii) Using the above identities show that
$[L_z, r^2] = 0$ where $r^2 = x^2 + y^2 + z^2$. [2]

(b) Using the relations

$$[\sigma_x, \sigma_y] = i\hbar\sigma_z \quad , \quad [\sigma_y, \sigma_z] = i\hbar\sigma_x \quad , \quad [\sigma_z, \sigma_x] = i\hbar\sigma_y$$

(i) Show that $[\sigma_i, \sigma^2] = 0$ where i = x , y, z and $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$. [6]
(ii) Consider an Hamiltonian H which has the following properties
$[H, \sigma^2] = 0$
$[H, \sigma_i] = 0$
Explain why the dynamical variable corresponding to only one component of the operator $\boldsymbol{\sigma}_i$ can be a measurable simultaneously with **H** and $\boldsymbol{\sigma}^2$. [3]

(c) If A and B are two operators which commute with their commutator [A,B], prove that $[A,B^3]=3B^2[A,B]$. [8]

Q.5.
(a) The Hamiltonian of a rotating system with moment of inertia I is given by the expression

$$H = \frac{1}{2I}(L_x^2 + L_y^2)$$

where $\vec{L} = \vec{r} \times \vec{p}$.

(i) Show that $y_l^m(\vartheta,\varphi)$ are eigenfunctions of H. [2]

(ii) Determine the eigen-value of H . [2]

(iii) Determine the degeneracy of the states for $l = 0,1,2$. [6]

(b) An electron is described by an Hamiltonian $H = H_0 + H_1$ where

$$H_0 = \frac{p^2}{2m} - \frac{e^2}{r}$$

and H_1 describes the contribution from an external force.

H_0 has the eigenfunctions $\psi_{nlm} = R_{nl}(r) Y_l^m(\vartheta,\varphi)$ and energy E_n .

Here

n = principal quantum number=1,2,3......,

l = orbital angular momentum quantum number= 0,1,2,3.......

m = projection of orbital angular momentum or z-component of l .

The eigen functions $\psi_{n\ell m}$ have following properties:

$$H_0 \psi_{n\ell m} = E_n \psi_{n\ell m}$$

$$H_1 \psi_{n\ell m} = \alpha \psi_{n\ell m}$$

A state of the electron is described by eigenfunction $\phi = N(\psi_{100} - \sqrt{2}\psi_{210})$

(i) Determine the normalization constant N. [5]

(ii) Determine the expectation value of H . [7]

(iii) Is φ an eigenfunction of H ? [3]

Note: $\int \psi^*_{n_1\ell_1 m_1} \psi_{n_2\ell_2 m_2} \, d\tau = \delta_{n_1 n_2} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$

@@@@END OF EXAMINATION@@@@

APPENDIX:

Given: h=6.62606876 x 10^{-34} J s,

$\hbar = 1.0546 \times 10^{-34}$ J s ,

c = velocity of light =2.99792 x 10^8 m s^{-1}

k=1.3807x 10^{-23} JK^{-1} .

1ev=1.6022 x 10^{-19} J. ,

mass of electron= 9.10938 x 10^{-31} kg.

mass of neutron/proton = 1.6749 x 10^{-27} kg ,

Useful Information:

[A , CD] = [A , C] D + C [A , D]

[A C , D] = A [C, D] + [A , D] C

$[\, r_i \,,\, p_j \,] = i\,\delta_{ij}$ where $r_i = (\,x\,,\,y\,,\,z\,)$ and $p_i = (\,p_x\,,\,p_y\,,\,p_z\,)$,
$[\, L_x \,,\, L_y \,] = i\hbar\; L_z$, $[\, L_y \,,\, L_z \,] = i\hbar\; L_x$, $[\, L_z \,,\, L_x \,] = i\hbar\; L_y$ where $\vec{L} = \vec{r} \times \vec{p}$,

The functions $Y_l^m\,(\vartheta,\,\varphi)$ are eigenfunctions of L^2 and L_z operators with the property

$$L^2\, Y_\ell^m(\vartheta,\varphi) = \ell(\ell+1)\,\hbar^2\, Y_\ell^m(\vartheta,\varphi)$$
$$L_z\, Y_\ell^m(\vartheta,\varphi) = m\,\hbar\, Y_\ell^m(\vartheta,\varphi)$$

Useful Integrals:

$$\int_{-\infty}^{\infty} dz\, e^{-\alpha z^2} = \sqrt{\frac{\pi}{\alpha}} \quad , \quad \int_{-\infty}^{\infty} dz\, z^2\, e^{-\alpha z^2} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} \quad , \quad \int_{-\infty}^{\infty} dz\, z^4\, e^{-\alpha z^2} = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$

$$\int_0^{\infty} exp(-t^2)dt = \frac{\sqrt{\pi}}{2} \quad , \quad \int_0^{\infty} t^{2n+1}\, exp(-at^2)\, dt = \frac{n\,!}{2\,a^{n+1}} \qquad \text{with Re a > 0, n =0,1,2,..}$$

$$\int_0^{\infty} t^{2n}\, exp(-at^2)\, dt = \frac{1.3.5......(2n-1)}{2^{n+1}\, a^n}\sqrt{\frac{\pi}{a}} \qquad \text{with Re a > 0 , n=0,1,2.....}$$

$$\int sin^2(x)dx = \frac{x}{2} - \frac{1}{4} sin(2x) \;,$$

$$\int sin(mx)\,sin(nx)\,dx = \frac{1}{2}\left[\frac{sin\{(m-n)x\}}{(m-n)} - \frac{sin\{(m+n)x\}}{(m+n)}\right]$$

$$\int sin(mx)\,cos(nx)dx = -\frac{1}{2}\left[\frac{cos[(m-n)x]}{(m-n)} + \frac{cos[(m+n)x]}{(m+n)}\right]$$

$$\int_{-\infty}^{\infty} H_n(\xi)\, H_m(\xi)\, exp(-\xi^2)\, d\xi = \pi^{\frac{1}{2}}\, 2^n\, n!\, \delta_{nm}$$ where H(ξ) are Hermite polynomials and are real.

$$\int_0^{\infty} t^{z-1} \exp(-kt)\, dt = k^{-z}\, \Gamma(z) \quad \mathrm{Re}\, z > 0, \mathrm{Re}\, k > 0. \,,$$

$$\Gamma(n+1) = n! \quad \textit{for } n = 1,2,\ldots \textit{ and } \Gamma(1) = 1.$$

$$\int x^n e^{-m\alpha x}\, dx = \frac{\Gamma(n+1)}{(m\alpha)^{n+1}} \qquad \textit{for } m > 0 \quad \textit{and } n \geq 0.$$

You can calculate the integrals you need by expressing powers of x through (repeated) differentiation with respect to the parameter in the exponential, e.g.

$$\int_a^b dx\; x \exp(-\gamma x) = -\frac{\partial}{\partial\gamma}\int_a^b dx \exp(-\gamma x) \;\; \textit{and} \;\; \int_a^b dx\; x^2 \exp(-\gamma x) = \frac{\partial^2}{\partial\gamma^2}\int_a^b dx \exp(-\gamma x) \;\; \textit{and so on.)}.$$