72

UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2010/2011

TITLE OF PAPER : COMPUTATIONAL METHODS I

COURSE NUMBER : P262

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS SIX PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THE PAPER UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR.

P262 Computational Methods I

Question one

Given the following non-homogeneous differential equation as

$$\frac{d^2 y(t)}{dt^2} - 6\frac{d\,y(t)}{dt} + 13\,y(t) = f(t)$$

(a) find its particular solution $y_p(t)$ if

(i) $f(t) = t^3 - 5t$, **(6 marks)**

(ii) $f(t) = 3\cos(t) - 4\sin(3t)$ and plot this particular solution for $t = 0$ to 20 . **(8 marks)**

(b) (i) find the general solution to the homogeneous part of the given equation $y_h(t)$ and then write down the general solution to the given non-homogeneous differential equation $y_g(t)$ **(4 marks)**

(ii) if the initial conditions are given as $y(0) = -5$ & $\left.\frac{d\,y(t)}{dt}\right|_{t=0} = 1$, then find its specific solution and plot it for $t = 0$ to 1 . **(7 marks)**

Question two

Given the following Legendre's differential equation as

$$\left(1-x^2\right)\frac{d^2\,y(x)}{d\,x^2}-2\,x\,\frac{d\,y(x)}{d\,x}+20\,y(x)=0 \quad ,$$

(a) (i) use *dsolve* command to find its general solution , **(2 marks)**

(ii) one of its independent solution is a polynomial and plot this polynomial independent solution for x = - 1 to + 1 . **(3 marks)**

(b) (i) set $y(x)=\sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$, utilize the power series method to find its two independent solutions up to n = 10 terms , **(18 marks)**

(ii) one of its independent solution is a polynomial and compare it with that in (a)(ii) and make a brief comment. **(2 marks)**

Question three

75

(a) Given the following matrix equation A X = b where

$$A = \begin{pmatrix} 2 & -5 & 2 \\ 0 & 3 & -1 \\ 3 & 1 & -2 \end{pmatrix} \quad , \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad b = \begin{pmatrix} -27 \\ 14 \\ -3 \end{pmatrix}$$

find A^{-1} (show details) and then use it to find the solution of X .

(8 marks)

(b) Given the following differential equations for a coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -7\, x_1(t) + 3\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 6\, x_1(t) - 10\, x_2(t) \end{cases}$$

(i) find the eigen frequencies ω and their respective eigen vectors of X .

Then write down the general solutions of $x_1(t)$ *and* $x_2(t)$ in terms of them. **(8 marks)**

(ii) if initial conditions are given as

$$x_1(0) = 3 \quad , \quad x_2(0) = -1 \quad , \quad \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} = 0 \quad and \quad \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} = 4 \; ,$$

find the specific solutions of $x_1(t)$ *and* $x_2(t)$. Plot both $x_1(t)$ *and* $x_2(t)$ for $t = 0$ *to* 5 and show them in a single display. **(9 marks)**

Question four

76

Given a one-dimensional wave equation as $\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u(x,t)}{\partial t^2}$,

(a) set $u(x,t) = F(x)\,G(t)$, use separation of variable scheme to deduce the following two ordinary differential equations :

$$\begin{cases} \dfrac{d^2 F(x)}{d x^2} = -k^2\, F(x) \\ \dfrac{d^2 G(t)}{d t^2} = -c^2\, k^2\, G(t) \end{cases}$$

(5 marks)

(b) The general solution of (a) can be written as

$$u(x,t) = \sum_{\forall k} u_k(x,t)$$
$$= \sum_{\forall k} \left(A_k \cos(k\,x) + B_k \sin(k\,x)\right)\left(C_k \cos(c\,k\,t) + D_k \sin(c\,k\,t)\right)$$

where A_k , B_k , C_k & D_k *are arbitrary constants*

(i) applying two fixed end conditions (i.e., $u_k(0,t) = 0 = u_k(L,t)$) and zero initial speed condition (i.e., $\left.\frac{\partial u_k(x,t)}{\partial t}\right|_{t=0} = 0$), deduce from the above general solution that $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{c\,n\pi t}{L}\right)$ where E_n (n = 1 , 2 , 3 , …..) are arbitrary constants. **(8 marks)**

(ii) If c = 3 , L = 7 and the initial position of the string is given as $u(x,0) = \begin{cases} 5\,x & \text{if } 0 \le x \le 2 \\ -2\,x + 14 & \text{if } 2 \le x \le 7 \end{cases}$

find the values of E_1 , E_2 , E_3 , ⋯⋯ , E_9 . Plot this truncated specific solution at t = 0 , t = 1 and t = 2 for the x range of 0 to 7 , show them in a single display. **(12 marks)**

Question five

(a) Given $\frac{d\,y(x)}{d\,x} = \frac{2\sqrt{y(x) - \ln(x)}}{x} + \frac{1}{x}$, $y(1) = 0$ & $x \geq 1$, its specific solution can be shown to be $y(x) = (\ln(x))^2 + \ln(x)$ and the exact value of y(2) is 1.173600195 .

(i) Use Euler's method , starting with x = 1 and taking $h = 0.2$ and do 5 steps , to find the approximate value of y at x = 2 . Compare this approximate value with the given exact value of y(2) and calculate their percentage difference. **(6 marks)**

(ii) Use Runge-Kutta method , starting with x = 1 and taking $h = 0.2$ and do 5 steps , to find the approximate value of y at x = 2 . Compare this approximate value with the given exact value of y(2) and calculate their percentage difference. **(7 marks)**

(b) Given the differential equation $\frac{d^2\,y(x)}{d\,x^2} = x\,\frac{d\,y(x)}{d\,x} - 5\,y(x) + 2\,x$ with initial conditions of $y(0) = 4$ & $\left.\frac{d\,y(x)}{d\,x}\right|_{x=0} = -2$,

(i) use *dsolve* command to find its specific solution of y(x) . Also find the exact value of y(1) . **(3 marks)**

(ii) Use Euler's method , starting with x = 0 and taking $h = 0.1$ and do 10 steps , to find the approximate value of y at x = 1 . Compare this approximate value with the exact value of y(1) obtained in (b)(i) and calculate their percentage difference. **(9 marks)**