UNIVERSITY OF SWAZILAND 84

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2010/2011

TITLE OF PAPER : MATHEMATICAL METHODS FOR PHYSICISTS

COURSE NUMBER : P272

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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P272 MATHEMATICAL METHODS FOR PHYSICIST

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Question one

(a) Given a scalar function in spherical coordinates as $f = r^3 \sin(\theta)\cos(\phi)$, find the value of $\vec{\nabla} f$ at a point $P(2, 120^0, 300^0)$. **(5 marks)**

(b) Given $\vec{F} = \vec{e}_x (y^2) + \vec{e}_y (x^2) + \vec{e}_z (-2xy)$ and find the value of

$\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (1, 2, 0)$, $P_2 : (3, 10, 0)$ and

(i) L : a straight line from P_1 to P_2 on x – y plane , i.e., z = 0 plane . **(7 marks)**

(ii) L : a parabolic path described by $y = x^2 + 1$ from P_1 to P_2 on x – y plane . Compare this answer with that obtained in (b)(i) and comment on whether the given $\vec{F}$ is a conservative vector field or not. **(8 marks)**

(iii) Find $\vec{\nabla} \times \vec{F}$. Does this result agree with the comment you made in (b)(ii)? **(5 marks)**

Question two

Given $\vec{F} = \vec{e}_\rho\left(\rho^2\right) + \vec{e}_\phi\left(2\rho^2 \sin\phi\right) + \vec{e}_z\left(3z^2\right)$ in cylindrical coordinates,

(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if S is the closed surface enclosing the cylindrical tube of cross-sectional radius 4 and tube height 5 , i.e., $S = S_1 + S_2 + S_3$ where

S_1 : $\left(z = 0\ ,\ 0 \le \rho \le 4\ ,\ 0 \le \phi \le 2\pi\ \ \&\ \ d\vec{s} = -\vec{e}_z\, \rho\, d\rho\, d\phi\right)$

S_2 : $\left(z = 5\ ,\ 0 \le \rho \le 4\ ,\ 0 \le \phi \le 2\pi\ \ \&\ \ d\vec{s} = +\vec{e}_z\, \rho\, d\rho\, d\phi\right)$

S_3 : $\left(\rho = 4\ ,\ 0 \le \phi \le 2\pi\ ,\ 0 \le z \le 5\ \ \&\ \ d\vec{s} = \vec{e}_\rho\, \rho\, d\phi\, dz \xrightarrow{\rho = 4} \vec{e}_\rho\, 4\, d\phi\, dz\right)$ **(12 marks)**

(b) (i) find $\vec{\nabla} \bullet \vec{F}$, **(4 marks)**

(ii) then evaluate the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in (a) , i.e., V : $0 \le \rho \le 4\ ,\ 0 \le \phi \le 2\pi\ ,\ 0 \le z \le 5\ \ \&\ \ dv = \rho\, d\rho\, d\phi\, dz$.

Compare this value with that obtained in (a) and make a brief comment.

(9 marks)

Question three

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Given the following non-homogeneous differential equation as

$$\frac{d^2 x(t)}{dt^2} - \frac{d x(t)}{dt} - 2\, x(t) = f(t) \quad ,$$

(a) find its particular solution $x_p(t)$ if

(i) $f(t) = 5\cos(2t)$, **(7 marks)**

(ii) $f(t) = 20\, e^{-3t}$, **(5 marks)**

(b) for the homogeneous part of the given non-homogeneous differential equation , i.e., $\frac{d^2 x(t)}{dt^2} - \frac{d x(t)}{dt} + 2\, x(t) = 0$, set $x(t) = e^{\alpha t}$ and find the appropriate values of α and thus write down its general solution $x_h(t)$ **(6 marks)**

(c) if $f(t) = 20\, e^{-3t}$, write down the general solution of the given non-homogeneous differential equation in terms of the answers obtained in (a) & (b) . If the initial conditions are $x(0) = 6$ & $\left.\frac{d x(t)}{dt}\right|_{t=0} = -1$, find its specific solution $x_s(t)$. **(7 marks)**

Question four

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(a) Given the following 2-D Laplace equation in spherical coordinates as

$$\nabla^2 f(r,\theta) = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r,\theta)}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f(r,\theta)}{\partial \theta}\right)$$

, set $f(r,\theta) = F(r)G(\theta)$ and use separation variable scheme to separate the above partial differential equation into two ordinary differential equations. **(5 marks)**

(b) Given a Legendre's differential equation as :

$$\left(1-x^2\right)\frac{d^2 y(x)}{dx^2} - 2x\frac{dy(x)}{dx} + 12\, y(x) = 0$$

, set $y(x) = \sum_{n=0}^{\infty} a_n x^{n+s}$ & $a_0 \neq 0$ and utilize the power series method,

(i) write down its indicial equations and show that $s = 0$ *or* 1 and $a_1 = 0$, **(8 marks)**

(ii) write down its recurrence relation. Set $a_0 = 1$ and use the recurrence relation to generate two independent solutions in power series form truncated up to a_6 term. Show that one of the independent solution is a polynomial. **(12 marks)**

Question five

(a) Given the following periodic function f(t) = f(t + T) = f(t + 2 T) = ... with the period T = 3 and its first period description is $f(t) = 2t \quad for \quad 0 \le t \le 3$ (i.e., a jigsaw shape periodic function), write down its Fourier series representation and calculate the values of the first three Fourier coefficients of both sine and cosine Fourier series. (i.e., the values of a_0 , a_1 , a_2 for cosine series and the values of b_1 , b_2 , b_3 for sine series).

(Hint :

$$\int_0^6 \sin\left(\frac{n\pi t}{3}\right)\sin\left(\frac{m\pi t}{3}\right)dt = \begin{cases} 0 & if \quad n \ne m \\ 3 & if \quad n = m = 1,2,3,\cdots \end{cases}$$

$$\int_0^6 \cos\left(\frac{n\pi t}{3}\right)\cos\left(\frac{m\pi t}{3}\right)dt = \begin{cases} 0 & if \quad n \ne m \\ 3 & if \quad n = m = 1,2,3,\cdots \\ 6 & if \quad n = m = 0 \end{cases}$$

$$\int_0^6 \sin\left(\frac{n\pi t}{3}\right)\cos\left(\frac{m\pi t}{3}\right)dt = 0$$

$$\int t \sin(at)\,dt = -\frac{1}{a}t\cos(at) + \frac{1}{a^2}\sin(at)$$

$$\int t \cos(at)\,dt = +\frac{1}{a}t\sin(at) + \frac{1}{a^2}\cos(at)$$

)

(10 marks)

(b) Given the following coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -6\,x_1(t) + 10\,x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 4\,x_1(t) - 12\,x_2(t) \end{cases}$$

(i) set $x_1(t) = X_1\,e^{i\omega t}$ & $x_2(t) = X_2\,e^{i\omega t}$, then the above given equations can be deduced to the following matrix equation $A\,X = -\omega^2\,X$ where

$$A = \begin{pmatrix} -6 & 10 \\ 4 & -12 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

(4 marks)

(ii) find the eigenfrequencies ω of the given coupled system , **(5 marks)**

(iii) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b)(ii), **(6 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

(u_1, u_2, u_3)	represents	(x, y, z)	for rectangular coordinate system
	represents	(ρ, ϕ, z)	for cylindrical coordinate system
	represents	(r, θ, ϕ)	for spherical coordinate system
$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$	represents	$(\vec{e}_x, \vec{e}_y, \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$	for spherical coordinate system
(h_1, h_2, h_3)	represents	$(1, 1, 1)$	for rectangular coordinate system
	represents	$(1, \rho, 1)$	for cylindrical coordinate system
	represents	$(1, r, r\sin(\theta))$	for spherical coordinate system