UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2010/2011

TITLE OF PAPER	:	MATHEMATICAL METHODS FOR PHYSICISTS
COURSE NUMBER	:	P272
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS SEVEN PAGES, INCLUDING THIS PAGE.

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Question one

(a) (i) Given P(- 4 , - 2 , - 7) in Cartesian coordinate system, find its cylindrical and spherical coordinates . **(5 marks)**

(ii) Given P(10 , 150^0 , 280^0) in spherical coordinate system, find its cylindrical and Cartesian coordinates . **(5 marks)**

(b) Express $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\rho$ & $\vec{e}_\phi$ in terms of $\vec{e}_x$, $\vec{e}_y$ & $\vec{e}_z$ and deduce that

$$\begin{cases} \dfrac{d\vec{e}_r}{dt} = \vec{e}_\theta \dfrac{d\theta}{dt} + \vec{e}_\phi \sin(\theta)\dfrac{d\phi}{dt} \\ \dfrac{d\vec{e}_\theta}{dt} = -\vec{e}_r \dfrac{d\theta}{dt} + \vec{e}_\phi \cos(\theta)\dfrac{d\phi}{dt} \\ \dfrac{d\vec{e}_\phi}{dt} = -\vec{e}_r \sin(\theta)\dfrac{d\phi}{dt} - \vec{e}_\theta \cos(\theta)\dfrac{d\phi}{dt} \end{cases}$$

(15 marks)

Question two

(a) Given a scalar function f in cylindrical system as $f = 5\rho^3 - 2\rho z^2 \cos(\phi)$,

(i) find $\vec{\nabla} f$ at the point $P(2, 200^0, -5)$. **(5 marks)**

(ii) show that $\vec{\nabla} \times (\vec{\nabla} f) = 0$. **(5 marks)**

(b) Given a vector field $\vec{F} = \vec{e}_r\, 8r^2 + \vec{e}_\theta\, r^2 \cos(\phi) + \vec{e}_\phi\, r^2 \sin(\theta)$ in spherical system,

(i) evaluate the value of the closed loop line integral $\oint_L \vec{F} \bullet d\vec{l}$ if L is a circular closed loop of radius 10 in counter clockwise sense on $\theta = 90^0$ *plane* (i.e., x – y plane) and centred at the origin, i.e.,

$r = 10$, $\theta = 90^0$, $0 \le \phi \le 2\pi$ & $d\vec{l} = \vec{e}_\phi\, r \sin(\theta)\, d\phi \xrightarrow{r=10\ ,\ \theta=90^0} \vec{e}_\phi\, 10\, d\phi$

(6 marks)

(ii) find $\vec{\nabla} \times \vec{F}$, then find the value of the surface integral $\iint_S (\vec{\nabla} \times \vec{F}) \bullet d\vec{s}$ where S is the semi-spherical surface of radius 10 enclosed by the given closed loop in (b)(i) , i.e.,

$r = 10$, $0 \le \theta \le 90^0$, $0 \le \phi \le 2\pi$ & $d\vec{s} = \vec{e}_r\, r^2 \sin(\theta)\, d\theta\, d\phi \xrightarrow{r=10} \vec{e}_r\, 100 \sin(\theta)\, d\theta\, d\phi$

Compare the result here with that obtained in (a) and make brief comment on Stokes' theorem. **(9 marks)**

Question three

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Given the following differential equation as :

$$\frac{d^2 y(x)}{dx^2} - \frac{d\, y(x)}{dx} - 2\, y(x) = 0$$

utilize the power series method , i.e., setting $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$,

(a) write down the indicial equations. Find the values of s and a_1 (setting $a_0 = 1$).

(10 marks)

(b) write down the recurrence relation. For all the appropriate values of s and a_1 found in (a), set $a_0 = 1$ and use the recurrence relation to calculate the values of a_n up to the value of a_5 . Thus write down two independent solutions in their power series forms.

(15 marks)

Question four

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An elastic string of length 10 is fixed at its two ends, i.e., at $x=0$ & $x=10$ and its transverse deflection $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=9\frac{\partial^2 u(x,t)}{\partial x^2},$$

(a) set $u(x,t)=F(x)\,G(t)$ and use separation scheme to deduce the following ordinary differential equations :

$$\begin{cases}\dfrac{d^2 F(x)}{d x^2}=-k^2 F(x)\\ \dfrac{d^2 G(t)}{d y^2}=-9k^2 G(t)\end{cases}\quad \textit{where } k \textit{ is a separation constant}$$

(4 marks)

(b) by direct substitution, show that . $u(x,t)=\sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right)\cos\left(\frac{3n\pi t}{10}\right)$.

where E_n $n=1,2,3,\cdots\cdots$ are arbitrary constants , satisfies two fixed end conditions , i.e., $u(0,t)=0=u(10,t)$, as well as zero initial speed condition , i.e., $\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=0$.

(6 marks)

(c) then find E_n in terms of n if the initial position of the string , i.e., $u(x,0)$, is given as $u(x,0)=\begin{cases}3x & \text{if } 0\le x\le 4\\ -2x+20 & \text{if } 4\le x\le 10\end{cases}$

(hint : $\int_{x=0}^{10}\sin\left(\frac{n\pi x}{10}\right)\sin\left(\frac{m\pi x}{10}\right)dx=\begin{cases}0 & \text{if } n\neq m\\ 5 & \text{if } n=m\end{cases}$ &

$\int x\sin\left(\frac{n\pi x}{10}\right)dx=\frac{100}{n^2\pi^2}\sin\left(\frac{n\pi x}{10}\right)-\frac{10}{n\pi}x\cos\left(\frac{n\pi x}{10}\right)$)

Thus calculate the values of E_1 , E_2 and E_3 . **(15 marks)**

Question five

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Given the following equations for coupled oscillator system as :

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -14\, x_1(t) + 4\, x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 5\, x_1(t) - 6\, x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation

$A\,X = -\,\omega^2\, X$ where $A = \begin{pmatrix} -14 & 4 \\ 5 & -6 \end{pmatrix}$ & $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ **(4 marks)**

(b) find the eigenfrequencies ω of the given coupled system , **(5 marks)**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b). **(6 marks)**

(d) find the normal coordinates for the given coupled system, **(7 marks)**

(e) write down the general solution of the given system. **(3 marks)**

Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\ \sin(\theta)\ \cos(\phi) \\ y = r\ \sin(\theta)\ \sin(\phi) \\ z = r\ \cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\ \cos(\phi) \\ y = \rho\ \sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1\, h_2\, h_3}\left(\frac{\partial(F_1\, h_2\, h_3)}{\partial u_1} + \frac{\partial(F_2\, h_1\, h_3)}{\partial u_2} + \frac{\partial(F_3\, h_1\, h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2\, h_3}\left(\frac{\partial(F_3\, h_3)}{\partial u_2} - \frac{\partial(F_2\, h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1\, h_3}\left(\frac{\partial(F_1\, h_1)}{\partial u_3} - \frac{\partial(F_3\, h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1\, h_2}\left(\frac{\partial(F_2\, h_2)}{\partial u_1} - \frac{\partial(F_1\, h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1\, F_1 + \vec{e}_2\, F_2 + \vec{e}_3\, F_3$ and

(u_1 , u_2 , u_3)	represents	(x , y , z)	for rectangular coordinate system
	represents	(ρ , ϕ , z)	for cylindrical coordinate system
	represents	(r , θ , ϕ)	for spherical coordinate system
$(\vec{e}_1 , \vec{e}_2 , \vec{e}_3)$	represents	$(\vec{e}_x , \vec{e}_y , \vec{e}_z)$	for rectangular coordinate system
	represents	$(\vec{e}_\rho , \vec{e}_\phi , \vec{e}_z)$	for cylindrical coordinate system
	represents	$(\vec{e}_r , \vec{e}_\theta , \vec{e}_\phi)$	for spherical coordinate system
(h_1 , h_2 , h_3)	represents	$(1 , 1 , 1)$	for rectangular coordinate system
	represents	$(1 , \rho , 1)$	for cylindrical coordinate system
	represents	$(1 , r , r\ \sin(\theta))$	for spherical coordinate system