UNIVERSITY OF SWAZILAND 131

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2010/2011

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|---|---|---|
| TITLE OF PAPER | : | ELECTROMAGNETIC THEORY I |
| COURSE NUMBER | : | P331 |
| TIME ALLOWED | : | THREE HOURS |
| INSTRUCTIONS | : | ANSWER ANY FOUR OUT OF FIVE QUESTIONS.<br>EACH QUESTION CARRIES 25 MARKS.<br><br>MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. |

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## Question one

(a) (i) From the conservation of electric charges, i.e.,

$$I = -\frac{\partial q}{\partial t} \quad where \quad I = \oint_S \vec{J} \bullet d\vec{s} \quad \& \quad q = \oint_V \rho_v \, dv \,,$$

deduce the following equation of continuity for electric charges

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

**( 4 marks )**

(ii) Show that without introducing the displacement current term, i.e., $\frac{\partial \vec{D}}{\partial t}$ , in the equation for Ampere's law, i.e., $\vec{\nabla} \times \vec{H} = \vec{J}$ instead of $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ , Maxwell's equations would contradict with the continuity equation for electric charges. And also show that with the introduction of the displacement current term $\frac{\partial \vec{D}}{\partial t}$ in the equation for Ampere's law, i.e., $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ , Maxwell's equations agree with the law of conservation of charges. **( 7 marks )**

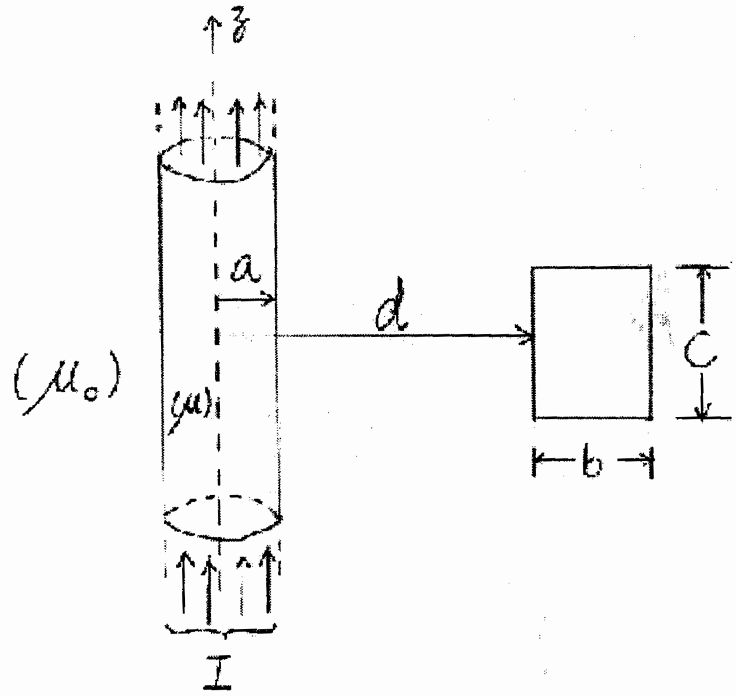
(b) A spherical ball of radius $R_0$ with a dielectric constant $\varepsilon$ , centered at the origin , carries a volume charge distribution of $\rho_v = 15\left(1+\alpha\, r^2\right) \frac{Coulomb}{m^3}$ where $\alpha$ is a constant and embedded in air of permittivity $\varepsilon_0$ .

(i) Use the integral form of Gauss's law and choose proper Gaussian surfaces to find $\vec{E}$ in terms of $r$ , $R_0$ & $\alpha$ for $0 \le r \le R_0$ & $r \ge R_0$ regions . **( 12 marks )**

(ii) Determine the value of $\alpha$ such that the electric field everywhere outside the spherical ball is zero. **( 2 marks )**

**Questin two** 

(a) A very long straight wire of cross-sectional radius $a$ and a permeability $\mu$, has its central axis coinciding with the z – axis. It carries an uniform current density in the positive z direction with the total static current of $I$, i.e., the current density inside the wire is $\frac{I}{\pi a^2}$, as shown in the diagram below :




Use the integral form of Ampere's law and choose proper closed loops to find the magnetic induction $\vec{B}$ in terms of $\rho$, $a$ & $I$ for $0 \le \rho \le a$ & $\rho \ge a$ regions, $\rho, \varphi, z$ are cylindrical coordinates. **( 15 marks )**

(b) Placing a rectangular conducting loop of dimension $b \times c$ a distance of $d\,(> a)$ away from the central axis of the wire as shown in the diagram in (a), i.e., the inner region confined by the rectangular loop in clockwise sense is

$S:\ d \le \rho \le d+b \ ,\ 0 \le z \le c\ \&\ d\vec{s} = \vec{a}_\phi\, d\rho\, dz$ .

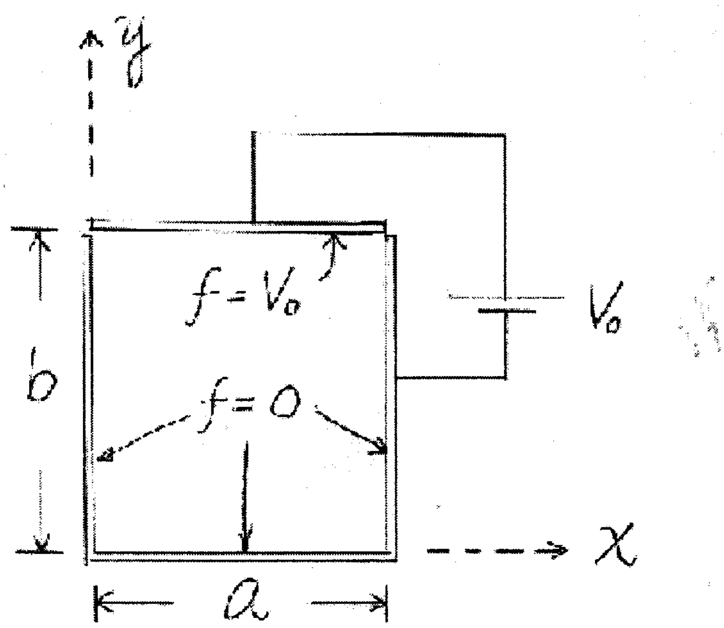
(i) Find the total magnetic flux passing through the inner region confined by the rectangular loop i.e., $\int_S \vec{B} \bullet d\vec{s}$ , in terms of $a, b, c, d\ \&\ I$ . **( 7 marks )**

(ii) If the wire carries a sinusoidal current of $I_0 \sin(\omega t)$ instead of carrying a static current $I$, find the induced e.m.f. in the rectangular conducting loop in terms of $a, b, c, d, \omega\ \&\ I_0$ under quasi static situation . **( 3 marks )**

## Question three



An infinitely long , rectangular U shaped conducting channel is insulated at the corners from the conducting plate forming the fourth side with interior dimensions as shown below :




The electric potential in between the two conductors $f(x,y)$ satisfies 2 – D Laplace equation , i.e., $\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 0$

(a) Using separation of variable scheme and setting $f(x,y) = X(x)\,Y(y)$, break the 2 – D Laplace equation into two ordinary differential equations . **( 5 marks )**

(b) By direct substitution, show that $f_n(x,y) = E_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$ , where $E_n$ *with* $n = 1,2,3,\cdots$ are constants , not only satisfies 2 – D Laplace equation but also satisfies the following three zero boundary conditions , i.e., $f_n = 0$ at $x = 0$ , $f_n = 0$ at $x = a$ & $f_n = 0$ at $y = 0$ . **( 8 marks )**

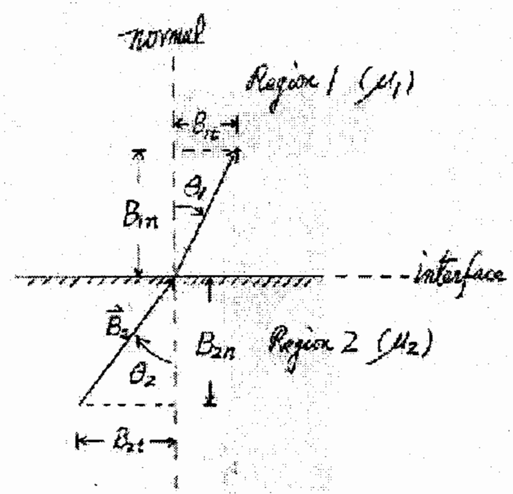
(c) Apply the final non-zero boundary condition , i.e., $f(x,b) = V_0$ , to the following $f(x,y)$ where $f(x,y) = \sum_{n=1}^{\infty} f_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$ to find $E_n$ in terms of $V_0$ , $a$ , $b$ & $n$ . **( 12 marks )**

(Hint : $\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) dx = \begin{cases} 0 & if \quad n \neq m \\ \frac{a}{2} & if \quad n = m \end{cases}$ )

## Question four



(a) An interface separating two isotropic materials of permeabilities $\mu_1$ & $\mu_2$ as shown in the diagram below. $\vec{B}_1$ & $\vec{B}_2$ are the magnetic fields at points on either side of the interface infinitely close to each other and $\theta_1$ & $\theta_2$ are the respective angles made with the normal.




(i) Use integral magnetic Gauss law and by choosing proper Gaussian surface (pillbox in shape across the interface) deduce that the normal component of $\vec{B}$ is continuous at the interface , i.e., $B_{1n} = B_{2n}$ . **( 6 marks )**

(ii) Given that the tangential component of $\vec{H}$ is continuous at the interface , i.e., $H_{1t} = H_{2t}$ , deduce the following refraction relation for $\vec{B}$

$$\tan(\theta_2) = \frac{\mu_2}{\mu_1} \tan(\theta_1)$$ **( 6 marks )**

(b) In a conductive region, based on Drude's model the force on a conduction electron is $-e\vec{E}$ and the retardation force due to the ion lattice of the conductor is $-\frac{2\, m_e\, \vec{v}_d}{\tau_c}$ .

where $\vec{v}_d$ & $\tau_c$ are the drifting velocity and mean free time of an average conduction electron. Thus the equation of motion for a conduction electron is

$$m_e \frac{d\,\vec{v}_d}{d\,t} = -e\,\vec{E} - \frac{2\, m_e\, \vec{v}_d}{\tau_c} \;,$$

(i) In the steady state situation , i.e., $\frac{d\,\vec{v}_d}{d\,t} = 0$ , deduce the following point form of Ohm's law $\vec{J} = \sigma\,\vec{E}$ *where* $\sigma = \frac{n\,e^2}{2\,m_e}\tau_c$ , (Hint : $\vec{J} = \rho_v\,\vec{v}_d = -n\,e\,\vec{v}_d$ ) **( 7 marks )**

(ii) If a certain pure metal having an atomic density of $2.5 \times 10^{28} \frac{atoms}{m^3}$ at room temperature and two outer orbit electrons are conduction electrons, find the value of $\tau_c$ if its measured dc conductivity is $\sigma = 1.4 \times 10^7 \frac{1}{m\,\Omega}$ . **( 6 marks )**

## Question five



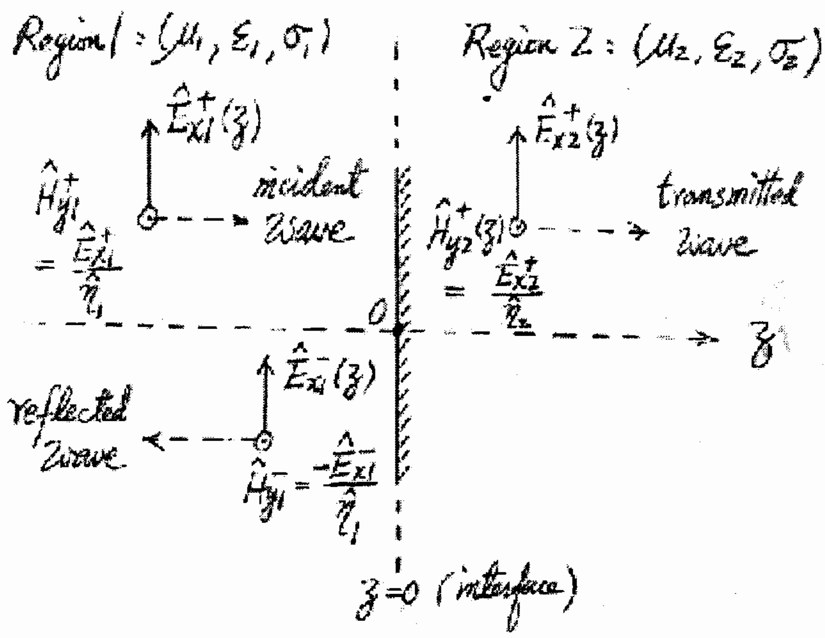
(a) An uniform plane wave traveling along the + z direction with the field components $E_x(z)$ & $H_y(z)$ has a complex electric field amplitude $\hat{E}_m = 100\, e^{i\,30^0}\ \frac{V}{m}$ and propagates at a frequency $f = 5\times10^7\ Hz$ in a material region having the parameters $\mu=\mu_0$ , $\varepsilon = 9\,\varepsilon_0$ & $\frac{\sigma}{\omega\,\varepsilon} = 0.5$ .

(i) Find the values of the propagation constant $\hat{\gamma}\ (=\alpha + i\,\beta)$ and the intrinsic wave impedance $\hat{\eta}$ for this wave , **( 4 marks )**

(ii) Express the electric and magnetic fields in both their complex and real-time forms , with the numerical values of (a)(i) inserted , **( 6 marks )**

(iii) Find the values of the penetration depth , wave length and phase velocity of the given wave . **( 3 marks )**

(b) An uniform plane wave is normally incident upon an interface separating two regions as shown below. The incident wave is given as $\left(\hat{E}^+_{x1} = \hat{E}^+_{m1}\, e^{-\hat{\gamma}_1 z} ,\ \hat{H}^+_{y1} = \frac{\hat{E}^+_{m1}}{\hat{\eta}_1}\, e^{-\hat{\gamma}_1 z}\right)$ and thus the reflected and transmitted wave can be written as $\left(\hat{E}^-_{x1} = \hat{E}^-_{m1}\, e^{+\hat{\gamma}_1 z} ,\ \hat{H}^-_{y1} = -\frac{\hat{E}^-_{m1}}{\hat{\eta}_1}\, e^{+\hat{\gamma}_1 z}\right)$ and $\left(\hat{E}^+_{x2} = \hat{E}^+_{m2}\, e^{-\hat{\gamma}_2 z} ,\ \hat{H}^+_{y2} = \frac{\hat{E}^+_{m2}}{\hat{\eta}_2}\, e^{-\hat{\gamma}_2 z}\right)$ respectively.



## Question five (continued)

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From the boundary conditions at the interface, i.e., both total $\hat{E}_x$ & $\hat{H}_y$ are continuous at z = 0, deduce the following

$$\begin{cases} \hat{E}^-_{m1} = \hat{E}^+_{m1} \dfrac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \\ \hat{E}^+_{m2} = \hat{E}^+_{m1} \dfrac{2\,\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_1} \end{cases}$$

**( 12 marks )**

## Useful informations

$e = 1.6 \times 10^{-19}\ C$

$m_e = 9.1 \times 10^{-31}\ kg$

$\mu_0 = 4\,\pi \times 10^{-7}\ \dfrac{H}{m}$

$\varepsilon_0 = 8.85 \times 10^{-12}\ \dfrac{F}{m}$

$\hat{\gamma} = \alpha + i\,\beta$ *where*

$$\alpha = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}-1}$$

$$\beta = \frac{\omega\sqrt{\mu\,\varepsilon}}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}+1}$$

$$\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = 3\times 10^8\ \frac{m}{s}$$

$$\hat{\eta} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\,\varepsilon}\right)^2}}\, e^{i\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\,\varepsilon}\right)}$$

$\eta_0 = 120\,\pi\ \ \Omega = 377\ \ \Omega$

$\beta_0 = \omega\sqrt{\mu_0\,\varepsilon_0}$

$$\oiint_S \vec{E}\bullet d\vec{s} = \frac{1}{\varepsilon}\iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B}\bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E}\bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B}\bullet d\vec{s}\right)$$

$$\oint_L \vec{B}\bullet d\vec{l} = \mu\iint_S \vec{J}\bullet d\vec{s} + \mu\,\varepsilon\frac{\partial}{\partial t}\left(\iint_S \vec{E}\bullet d\vec{s}\right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma \vec{E}$$

$$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d v$$ *divergence theorem*

$$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$$ *Stokes' theorem*

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho} \frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$

$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial (F_x)}{\partial x} + \frac{\partial (F_y)}{\partial y} + \frac{\partial (F_z)}{\partial z} = \frac{1}{\rho} \frac{\partial (F_\rho\, \rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial (F_\phi)}{\partial \phi} + \frac{\partial (F_z)}{\partial z}$$

$$= \frac{1}{r^2} \frac{\partial (F_r\, r^2)}{\partial r} + \frac{1}{r \sin(\theta)} \frac{\partial (F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r \sin(\theta)} \frac{\partial (F_\phi)}{\partial \phi}$$

$$\vec{\nabla} \times \vec{F} = \vec{e}_x \left(\frac{\partial (F_z)}{\partial y} - \frac{\partial (F_y)}{\partial z}\right) + \vec{e}_y \left(\frac{\partial (F_x)}{\partial z} - \frac{\partial (F_z)}{\partial x}\right) + \vec{e}_z \left(\frac{\partial (F_y)}{\partial x} - \frac{\partial (F_x)}{\partial y}\right)$$

$$= \frac{\vec{e}_\rho}{\rho} \left(\frac{\partial (F_z)}{\partial \phi} - \frac{\partial (F_\phi\, \rho)}{\partial z}\right) + \vec{e}_\phi \left(\frac{\partial (F_\rho)}{\partial z} - \frac{\partial (F_z)}{\partial \rho}\right) + \frac{\vec{e}_z}{\rho} \left(\frac{\partial (F_\phi\, \rho)}{\partial \rho} - \frac{\partial (F_\rho)}{\partial \phi}\right)$$

$$= \frac{\vec{e}_r}{r^2 \sin(\theta)} \left(\frac{\partial (F_\phi\, r \sin(\theta))}{\partial \theta} - \frac{\partial (F_\theta\, r)}{\partial \phi}\right) + \frac{\vec{e}_\theta}{r \sin(\theta)} \left(\frac{\partial (F_r)}{\partial \phi} - \frac{\partial (F_\phi\, r \sin(\theta))}{\partial r}\right) + \frac{\vec{e}_\phi}{r} \left(\frac{\partial (F_\theta\, r)}{\partial r} - \frac{\partial (F_r)}{\partial \theta}\right)$$

where $\vec{F} = \vec{e}_x F_x + \vec{e}_y F_y + \vec{e}_z F_z = \vec{e}_\rho F_\rho + \vec{e}_\phi F_\phi + \vec{e}_z F_z = \vec{e}_r F_r + \vec{e}_\theta F_\theta + \vec{e}_\phi F_\phi$ and

$$d\vec{l} = \vec{e}_x\, dx + \vec{e}_y\, dy + \vec{e}_z\, dz = \vec{e}_\rho\, d\rho + \vec{e}_\phi\, \rho\, d\phi + \vec{e}_z\, dz = \vec{e}_r\, dr + \vec{e}_\theta\, r\, d\theta + \vec{e}_\phi\, r \sin(\theta)\, d\phi$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2}$$