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UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION: 2010/2011

TITLE OF THE PAPER: COMPUTATIONAL METHODS-II

COURSE NUMBER: P482

TIME ALLOWED:
SECTION A: ONE HOUR
SECTION B: TWO HOURS

INSTRUCTIONS:
THE ARE TWO SECTIONS IN THIS PAPER:

- **SECTION A** IS A WRITTEN PART. ANSWER THIS SECTION ON THE ANSWER BOOK. IT CARRIES A TOTAL OF **30** MARKS.
- **SECTION B** IS A PRACTICAL PART WHICH YOU WILL WORK ON A PC AND SUBMIT THE PRINTED OUTPUT. IT CARRIES A TOTAL OF **70** MARKS.

Answer **any** two questions from **section A** and
all the questions from **section B**.
Marks for different sections of each Question are shown in the right hand margin.

THE PAPER HAS 8 PAGES, INCLUDING THIS PAGE.

DO NOT OPEN THIS PAGE UNTIL PERMISSION HAS BEEN GIVEN BY THE INVIGILATOR

Section A

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Question 1

(a) One of the simplest uniform random number generators has its sequence of numbers given by

$$x_{i+1} = (ax_i + b) mod\, c$$

where a, b, and c are magic numbers.

(i) State two most important criteria for a good uniform random number generator.

[2 mark]

(ii) One option of random numbers could be: $a = 7^5$, $b = 0$, and $c = 2^{31} - 1$. What is the period of the number generator in this case?

[2 mark]

(b) A uniform random number generator is used to generate a random sequence $[r_i, i = 1..N]$ with numbers that are uniformly distributed between 0 and 1.

(i) Show that the k^{th} moment of the sequence which defined as

$$\frac{1}{N}\sum_{i=1}^{N} r_i^k \cong \frac{1}{k+1}$$

[4 marks]

(ii) Discuss another test that can be used to test the randomness or the uniformity of the random sequence.

[2 marks]

(c) What is the range of the sequence generated by the built-in uniform random number generator in Maple.

[1 mark]

(d) A simulation for coin flipping. Write down an algorithm that generates a list that contains the outcome of 20 flips of an unbiased coin using the Maple random number generator.

[4 mark]

Question 2

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(a) Write down the finite difference algorithms that can be use to find approximate solutions of the following equations. Express your answer in terms discrete coordinates, for example $V(x)$ can be written as $V[i] = V[x_i]$, where $x_i = i\Delta x$ and Δx is the step-size.

(i) $\frac{\partial^2 V(x)}{\partial x^2} = 2\pi\rho(x)$

(ii) $\frac{\partial P(x,t)}{\partial t} + v_0 \frac{\partial P(x,t)}{\partial x} = 0$

(iii) $\frac{\partial \phi(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial x} + \frac{\partial^2 \phi(x,t)}{\partial x^2}$

[9 marks]

(b) The dynamics of a parachutist is described by the differential equation

$$m\frac{d^2y(t)}{dt^2} = gy(t) - b\frac{dy(t)}{dt} - c\left(\frac{dy(t)}{dt}\right)^2$$

where g is the acceleration due to gravity, b and c are phenomenological constants that describe the effects of the air drag. Show how to re-express this equation as a system of first order differential equation that can be solve numerically, for example using the Euler method.

[2 marks]

(c) Derive an expression for the truncation error of the following difference approximation.

$$\left(\frac{dy(x)}{dx}\right) = \frac{y_{j+1} - y_{j-1}}{2\Delta x}$$

[4 marks]

Question 3

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(a) The basic law of nature for spontaneous radioactive decay is that the number of decays ΔN in a time interval Δt is proportional to the number of particles $N(t)$ present at that time and to the time interval

$$\frac{\Delta N(t)}{\Delta t} = -\lambda N(t)$$

here $\lambda > 0$ is the decay constant. The decay radioactive nuclei is a stochastic process, which means that there is an element of chance involved in just when a given nucleus will decay, and so no two experiments are expected to give exactly the same results.

(i) Utilizing the above equation, explain how a process that is spontaneous and random at its very heart can lead to exponential decay.

[6 marks]

(ii) How can you graphical show that an exponential law $N(t) = N_0 e^{-\lambda t}$ is different to a power law such as $N(t) = N_0 t^{-\lambda}$?

[2 marks]

(b) In the two dimensional Ising model for magnetic systems, the magnetic spin at site (i, j) is given by $S[i, j] = \pm 1$. The plus represents a spin-up and the minus represents a spin-down. In the paramagnetic state (non-magnetic state), the spins' orientation at the lattice sites is random. Write down algorithm that generates the paramagnetic state for the Ising model on a lattice with 128×128 lattice points.

[7 marks]

Section B

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Question 4

Spectral analysis of signal. Calculate and plot the power spectrum of the following signal:

$$R(t) = \sin(\omega_0 t) + \sin(\omega_1 t).$$

where $\omega_0 = 3.0$ rads/s and $\omega_1 = 0.5$ rads/s.
You may need to discretize the variable t into $t_i = i\Delta t$, where $i = 1, 2, 3....N$ and Δt is the time-step. You may take $\Delta t = 1$ and $N = 256$. In this case, the power spectrum of the function $R(t_i)$ is given by

$$P(\omega_i) = |r(\omega_i)|^2$$

where $r(\omega_i)$ is the Fourier transform of $R(t_i)$, and $\omega_i = 2\pi i/(N\Delta t)$. Since the FFT procedure returns $r(\omega_i)$ for $i = -N/2$ to $N/2 - 1$. Plot $P(\omega_i)$ for the positive ω_i values only.

[15 marks]

Question 5

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Heavy nuclides like ^{235}U are unstable and decay into lighter nuclei under the emission of particles such as Helium (α - radiation), electrons (β-particle), and photons (γ-rays). The decay of a material containing $N(t)$ radioactive nuclei is statistically described by the model

$$\frac{dN(t)}{dt} = -\lambda N(t)$$

where τ is the time constant. Given that $\lambda = 2.6 \times 10^8 s^{-1}$, and the initial value of radioactive nuclei $N(0) = 80$nuclei.

(a) Calculate N(t) using the built-in Euler algorithm in Maple with the stepsize $dt = 1ns$, where $1ns = 1 \times 10^{-9}s$. Plot N(t) vs t, for $t = 0..10ns$. Label your graph properly.

[8 marks]

(b) What is the number of radioactive nuclei at $t = 14ns$.

[2 marks]

Question 6

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An $10cm$ long aluminium bar is insulated along its length with one end kept at $0K$ and the other end at $100K$. Assume that the initial the bar was everywhere $T(t = 0, x) = 80K$. The thermal conductivity, specific heat and density for Al are

$$\sigma = 237M/mK, C = 900J/kgK, \rho = 2700kg/m^3$$

The time and spatial dependent temperature profile is described by the heat equation

$$\frac{\partial T(t,x)}{\partial t} = \frac{\sigma}{C\rho}\frac{\partial^2 T(t,x)}{\partial x^2}$$

The numerical solution of the equation can be obtained numerically using the program *heat.mw* given below

Inputs:

```
sigma:=237000;
C:=900;
rho:=2700;
Df:=sigma/(rho*C);
dx:=1/10! step size
Nd:=10/dx! number of meshpoints
dt:=1/100! time step
Nt:=5/dt I number of iterations
```

Initial conditions

```
for i from 1 to Nd do
T[0,i]:=80;
end do:
```

Boundary condtions

```
for n from 0 to Nt do
T[n,0]:=0;
T[n,Nd]:=100;
end do:
```

Implementation stage

```
for n from 0 to Nt do
for i from 1 to Nd-1 do
T[n+1,i]:= T[n,i]+Df*dt*(T[n,i+1]-2*T[n,i]+T[n,i-1])/(dx*dx)
end do:
end do:
```

In this program the step size $dx = 0.1cm$ and the time step $dt = 0.01s$

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(a) Write a program or modify *heat.mw* to determine how the temperature varies with time and location. On one graph plot the temperature profiles: $T(t = 0, x)$, $T(t = 1s, x)$, and $T(t = 5s, x)$. Label your axis properly.

[15 marks]

(b) Find the average temperature of the bar at $t = 5s$.

[5 marks]

Imagine now that instead of being insulated along its length, the Al bar is in contact with an environment at a temperature $T_e = 80K$. Newton's law of cooling (radiation) says that the rate of temperature is

$$\frac{\partial T(t,x)}{\partial t} = -h(T(t,x) - T_e).$$

where h is a positive constant. This leads to a modified heat equation.

$$\frac{\partial T(t,x)}{\partial t} = \frac{\sigma}{C\rho}\frac{\partial^2 T(t,x)}{\partial x^2} - h(T(t,x) - T_e).$$

Now modify *heat.mw* to include Newton's.

(c) Given $h = 0.5s^{-1}$ determine the temperature of the bar when the Newton cooling mechanism is included. On one graph plot the temperature profiles: $T(t = 0, x)$, $T(t = 1s, x)$, and $T(t = 5s, x)$.

[20 marks]

(d) Calculate the average temperature of the bar at $t = 5s$ and discuss the effects of Newton cooling.

[5 marks]