UNIVERSITY OF SWAZILAND

82

FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2011/2012

TITLE OF PAPER	:	COMPUTATIONAL METHODS I
COURSE NUMBER	:	P262
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS SIX PAGES, INCLUDING THIS PAGE.

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P262 Computational Methods I

Question one

Given the following non-homogeneous ordinary differential equation as

$$\frac{d^2\, y(t)}{d\,t^2} + 2\,\frac{d\, y(t)}{d\,t} + 5\; y(t) = 10\,\sin(t) + 13\,\cos(3t)$$

(a) find its particular solution $y_p(t)$, **(9 marks)**

(b) find the general solution $y_h(t)$ for the homogeneous part of the given differential equation, **(4 marks)**

(c) find the general solution $y_g(t)$ for the above given non-homogeneous differential equation, **(2 marks)**

(d) if given initial conditions as $y(0) = 9$ *and* $\left.\frac{d\, y(t)}{d\,t}\right|_{t=0} = 1$, find its specific solution of $y(t)$, i.e., $y_s(t)$. Plot $y_s(t)$ for $t = 0$ *to* 30 and make a brief comment on its large t behavior. **(10 marks)**

Question two

Given the following differential equation as

$$\frac{d^2\, y(x)}{d\,x^2} + 6\,\frac{d\,y(t)}{d\,t} + 8\;y(t) = 0$$

set $\quad y(x) = \sum_{n=0}^{\infty} a_n\; x^{n+s} \qquad and \qquad a_0 \neq 0$, utilize the power series method and

(a) write down the indicial equations and find the values of s and possibly the value of a_1 (if a_1 is in terms of a_0 and s , then find the possible values of a_1 by setting $a_0 = 1$) **(7 marks)**

(b) write down the recurrence relation. Set $a_0 = 1$ and use the recurrence relation to find the values of a_n ($n = 2 \;\; to \;\; 10$) for each value of s found in (a). Write down two independent series solutions truncated up to a_{10} term.

(8 marks)

(c) (i) write the general solution for the above given differential equation,

(2 marks)

(ii) if given initial conditions as $y(0) = 3 \;\; and \;\; \left.\frac{d\,y(x)}{d\,x}\right|_{x=0} = -1$, find the specific solution and plot it for $x = 0 \;\; to \;\; 1$. **(8 marks)**

Question three

Given the following differential equations for a coupled oscillator system as

$$\begin{cases} \dfrac{d^2 x_1(t)}{dt^2} = -4\,x_1(t) + 3\,x_2(t) \\ \dfrac{d^2 x_2(t)}{dt^2} = 2\,x_1(t) - 5\,x_2(t) \end{cases}$$

(a) set $x_1(t) = X_1\, e^{i\omega t}$ *and* $x_2(t) = X_2\, e^{i\omega t}$, deduce the following matrix equation $A\,X = -\,\omega^2\,X$ where

$A = \begin{pmatrix} -4 & 3 \\ 2 & -5 \end{pmatrix}$ *and* $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, **(4 marks)**

(b) (i) find the eigen frequencies of ω , **(4 marks)**

(ii) find the eigen vectors of X , **(4 marks)**

(c) (i) write down the general solutions of $x_1(t)$ *and* $x_2(t)$ in terms of the eigenfrequencies and eigenvectors obtained in (b) , **(4 marks)**

(ii) if initial conditions are given as

$$x_1(0) = 2 \;,\quad x_2(0) = -4 \;,\quad \left.\frac{d\,x_1(t)}{dt}\right|_{t=0} = -1 \quad and \quad \left.\frac{d\,x_2(t)}{dt}\right|_{t=0} = 1 \;,$$

find the specific solutions of $x_1(t)$ *and* $x_2(t)$. Plot both $x_1(t)$ *and* $x_2(t)$ for $t = 0$ *to* 10 and show them in a single display. **(9 marks)**

Question four

(a) Given a scalar function $f = 5\,x\,y^2 + 2\,y\,z^2 - 3\,x\,y\,z$,

(i) find the value of $\vec{\nabla} f$ at the point P : (1 , - 1 , 2) , **(3 marks)**

(ii) find the directional derivative of f at the point P : (1 , - 1 , 2) along the direction of $\left(\vec{e}_x\, 2 - \vec{e}_y\, 3 + \vec{e}_z\right)$, i.e., [2 , - 3 , 1] . **(4 marks)**

(b) Given a vector field $\vec{F} = \vec{e}_x\left(5\,y^2\right) + \vec{e}_y\left(10\,x\,y - 6\,z^2\right) + \vec{e}_z\left(-12\,y\,z\right)$, i.e.,

$\vec{F}$ = [$5\,y^2$, $10\,x\,y - 6\,z^2$, $- 12\,y\,z$] , find the value of the line integral of $\vec{F}$ from the point P_1 : (1 , 6 , 0) to the point P_2 : (3 , 2 , 0) along a line path of L , i.e., $\int_{P_1,L}^{P_2} \vec{F} \bullet d\vec{l}$,

(i) if L : a straight line from P_1 to P_2 on z = 0 plane , **(8 marks)**

(ii) if L : a hyperbolic path described by $y = \dfrac{6}{x}$ from P_1 to P_2 on z = 0 plane . Compare this answer with that obtained in (b)(i) and comment on whether the given $\vec{F}$ is a conservative vector field or not., **(7 marks)**

(iii) use *potential* command to find out whether the given $\vec{F}$ is a conservative vector field or not. If yes, then find its associated scalar potential. **(3 marks)**

Question five

One-dimensional wave equation for a vibrating elastic string of length L can be written as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$ where $u(x,t)$ is a longitudinal vibration amplitude function

and c is a constant.

(a) The general solution of the given partial differential equation can be written as

$$u(x,t) = \sum_{\forall k} u_k(x,t)$$
$$= \sum_{\forall k} (A_k \cos(kx) + B_k \sin(kx))(C_k \cos(ckx) + D_k \sin(ckx))$$

where A_k , B_k , C_k & D_k are arbitrary constants.

Applying two fixed end conditions (i.e., $u_k(0,t) = 0 = u_k(L,t)$) and zero initial speed condition (i.e., $\left.\frac{\partial u_k(x,t)}{\partial t}\right|_{t=0} = 0$), deduce from the above general solution that $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right)$

where E_n (n = 1 , 2 , 3 , ...) are arbitrary constants. **(8 marks)**

(b) if c = 5 , L = 10 and the initial position of the string is given as

$$u(x,0) = \begin{cases} 3x & if \quad 0 \le x \le 4 \\ -2x + 20 & if \quad 4 \le x \le 10 \end{cases}$$

(i) find the values of E_1 , E_2 , E_3 , $\cdots\cdots$, E_{10} . Write down the specific solution of u(x,t) in its series expression up to E_{10} term.

(11 marks)

(ii) plot the solution obtained in (b)(i) at t = 0 , t = 1 and t = 2 respectively, i.e., u(x,0) , u(x,1) and u(x,2) , for the range of x values from x = 0 to x = 10. Show them in a single display. **(6 marks)**