UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2011/2012

TITLE OF PAPER : COMPUTATIONAL METHODS I

COURSE NUMBER : P262

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF SIX QUESTIONS. EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

STUDENTS ARE PERMITTED TO USE MAPLE TO ANSWER THE QUESTIONS.

THIS PAPER HAS SEVEN PAGES, INCLUDING THIS PAGE.

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P262 Computational Methods I

Question one

(a) Given the following 2^{nd} order homogeneous differential equation as

$$\frac{d^2 y(x)}{dx^2} + 2\frac{d\,y(x)}{dx} + 5\,y(x) = 0$$

(i) set $y(x) = e^{ax}$ and find the appropriate values of a . Write down its general solution. **(4 marks)**

(ii) if the initial conditions are given as $y(0) = -1$ & $\left.\frac{d\,y(x)}{dx}\right|_{x=0} = 2$,

then find its specific solution and plot it for x = 0 to 5 . **(5 marks)**

(b) Given the following non-homogeneous differential equation as

$$\frac{d^2 y(t)}{dt^2} + 4\frac{d\,y(t)}{dt} + 10\,y(t) = 5\,e^{-4t} - 2\,t$$

(i) find its particular solution $y_p(t)$, **(6 marks)**

(ii) find the general solution to the homogeneous part of the given equation $y_h(t)$ and then write down the general solution to the given non-homogeneous differential equation $y_g(t)$ **(5 marks)**

(iii) if the initial conditions are given as $y(0) = 4$ & $\left.\frac{d\,y(t)}{dt}\right|_{t=0} = -1$,

then find its specific solution and plot it for x = 0 to 5 . **(5 marks)**

Question two

Given the following Bessel's equation as

$$x^2 \frac{d^2\, y(x)}{d\,x^2} + x \frac{d\, y(x)}{d\,x} + \left(x^2 - 4\right) y(x) = 0 \quad ,$$

(a) (i) use *dsolve* command to find its general solution , **(2 marks)**

(ii) use *series* command to express BesselJ(2,x) & BesselY(2,x) into their power series up to x^{11} (i.e., would appear with $0\left(x^{12}\right)$).
Then convert them into polynomials. **(4 marks)**

(b) (i) set $y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+s}$ *and* $a_0 \neq 0$, utilize the power series method to find its indicial equations and thus find the values of s & a_1 , **(6 marks)**

(ii) for $s = 2$, set $a_0 = 1$, use the recurrence relation to find the values of a_n up to $n = 8$. Show that this polynomial solution is linearly dependent to the independent solutions in (a)(ii). **(9 marks)**

(iii) for $s = -2$, set $a_0 = 1$, use the recurrence relation to find the values of a_n up to $n = 8$. Show that this polynomial solution can not be found directly by power series method. **(4 marks)**

Question three

(a) Given the following system of linear equations as :

$$\begin{cases} -10\,x_1 + 5\,x_2 - 8\,x_3 = 41 \\ 7\,x_1 - 3\,x_2 + 6\,x_3 = -28 \\ 6\,x_1 - 3\,x_2 + x_3 = -17 \end{cases}$$

(i) solve them by Gauss elimination , **(4 marks)**

(ii) solve them by Crammer's rule . **(4 marks)**

(b) Given the following system of first order differential equations as :

$$\begin{cases} \dfrac{d\,x_1(t)}{d\,t} = 9\,x_1(t) - 3\,x_2(t) \\ \dfrac{d\,x_2(t)}{d\,t} = 4\,x_1(t) - 4\,x_2(t) \end{cases}$$

(i) Set $x_1(t) = X_1\,e^{\lambda t}$ & $x_2(t) = X_2\,e^{\lambda t}$ and deduce the following matrix equation $A\,X = \lambda\,X$, where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. **(4 marks)**

(ii) Find the eigenvalues λ . For each eigenvalue find its eigenvector. **(4 marks)**

(iii) Write down the general solutions of $x_1(t)$ & $x_2(t)$. **(2 marks)**

(iv) If the following initial conditions are given as $x_1(0) = 3$ & $x_2(0) = -2$, find the specific solutions of $x_1(t)$ & $x_2(t)$. Plot these $x_1(t)$ & $x_2(t)$ for t from 0 to 1 and show them in a single display . **(7 marks)**

Question four

Given a vector field as $\vec{F} = \vec{e}_x\, 2\,x\,z + \vec{e}_y\, 5\,x\,y + \vec{e}_z\, y^2$,

(a) find the value of $\int_S \vec{F} \bullet d\vec{s}$ if the surface S is given as :

$S:\ 4\,x^2 + y^2 = 4 \quad , \quad 1 \le z \le 5$

(Hint : set $x = \cos(t)$ & $y = 2\sin(t)$ *where* $0 \le t \le 2\pi$) **(10 marks)**

(b) utilize the Divergence theorem , i.e., $\oint\!\!\int_S \vec{F} \bullet d\vec{s} \equiv \iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) d\,v$, and find the value of $\oint\!\!\int_S \vec{F} \bullet d\vec{s}$ if the closed surface S is the cover surface of a box with $0 \le x \le 1\,, 0 \le y \le 2\ \&\ 0 \le z \le 3$, **(7 marks)**

(c) use the given $\vec{F}$ to show that it satisfies the following vector identity :
$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \vec{e}_x\left(\nabla^2 F_x\right) - \vec{e}_y\left(\nabla^2 F_y\right) - \vec{e}_z\left(\nabla^2 F_z\right)$. **(8 marks)**

Question five

Given the following non-homogeneous differential equation as :

$$\frac{d^2 y(t)}{dt^2} - 3\frac{d\,y(t)}{dt} + 2\,y(t) = f(t)$$

where $f(t)$ is a periodic function with its period = 2 , i.e.,
$f(t) = f(t+2) = f(t+4) = f(t+6) = \cdots\cdots$, and its first period behaviour is given as

$$f(t) = \begin{cases} t & if \quad 0 \le t \le 1 \\ -t+2 & if \quad 1 \le t \le 2 \end{cases} ,$$

(a) (i) find the Fourier series representation of $f(t)$ up to n = 10 and name this truncated series as $f_{10}(t)$, **(7 marks)**

(ii) find the particular solution of $y(t)$ corresponding to $f_{10}(t)$ replacing $f(t)$ in the given non-homogeneous differential equation , **(9 marks)**

(b) (i) find the general solution for the homogeneous part of the given differential equation , i.e., $\frac{d^2 y(t)}{dt^2} - 3\frac{d\,y(t)}{dt} + 2\,y(t) = 0$, then write down the general solution for the given non-homogeneous differential equation , **(4 marks)**

(ii) find the specific solution to the given non-homogeneous differential equation if the initial conditions are given as

$$y(0) = -5 \quad \& \quad \left.\frac{d\,y(t)}{dt}\right|_{t=0} = 2 .$$ **(5 marks)**

Question six

A vibrating string of length L is fixed at its two ends , i.e., x = 0 & x = L . Its transverse displacement $u(x,t)$ satisfies the following one-dimensional wave equation

$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$ where c is a constant related to the properties of the given string ,

(a) set $u(x,y) = F(x)\,G(y)$ and utilize the separation of variable scheme to break the above partitial differential equation into two ordinary differential equations.

(4 marks)

(b) The general solution of the above partitial differential equation can be written as

$$u(x,t) = \sum_{\forall k} u_k(x,t)$$
$$= \sum_{\forall k} \left(A_k \cos(k\,x) + B_k \sin(k\,x)\right)\left(C_k \cos(c\,k\,t) + D_k \sin(c\,k\,t)\right)$$

where A_k , B_k , C_k & D_k are arbitrary constants.

(i) Applying two fixed end conditions, i.e., $u_k(0,t) = 0$ & $u_k(L,t) = 0$ and one zero initial speed condition , i.e., $\left.\frac{\partial u_k(x,t)}{\partial t}\right|_{t=0} = 0$, show that the above general solution can be deduced to

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{c n \pi t}{L}\right)$$ where E_n (n = 1 , 2 , 3 , ...) are arbitrary constants.

(8 marks)

(ii) If c = 3 , L = 10 and the initial position of the string is given as

$$u(x,0) = \begin{cases} 3x & \text{if} \quad 0 \le x \le 2 \\ 6 & \text{if} \quad 2 \le x \le 7 \\ -x + 10 & \text{if} \quad 7 \le x \le 10 \end{cases}$$

find the values of $E_1 , E_2 , E_3 , \cdots , E_6$. Then plot this specific polynomial solutions of t = 0 , t = 0.3 and t = 0.6 all for the same range of $x = 0$ *to* 10 and show them in a single display.

(13 marks)