UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2011/2012

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| TITLE OF PAPER | : | MATHEMATICAL METHODS FOR PHYSICISTS |
| COURSE NUMBER | : | P272 |
| TIME ALLOWED | : | THREE HOURS |
| INSTRUCTIONS | : | ANSWER ANY FOUR OUT OF FIVE QUESTIONS.<br>EACH QUESTION CARRIES 25 MARKS.<br><br>MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. |

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## P272 MATHEMATICAL METHODS FOR PHYSICIST

**Question one**

(a) Given an arbitrary scalar and continuous function in cylindrical coordinates as $f(\rho, \phi, z)$, prove that $\vec{\nabla} \times \left(\vec{\nabla} f(\rho, \phi, z)\right) \equiv 0$ . **( 6 marks )**

(b) Given $\vec{F} = \vec{e}_x \left(3 y^2\right) + \vec{e}_y \left(2 x z\right) + \vec{e}_z \left(6 x^2\right)$ and find the value of the following line integral

$\int_{P_1, L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (1, 1, 3)$ , $P_2 : (3, 9, 3)$ and

(i) L : a straight line from $P_1$ to $P_2$ on $z = 3$ plane . **( 7 marks )**

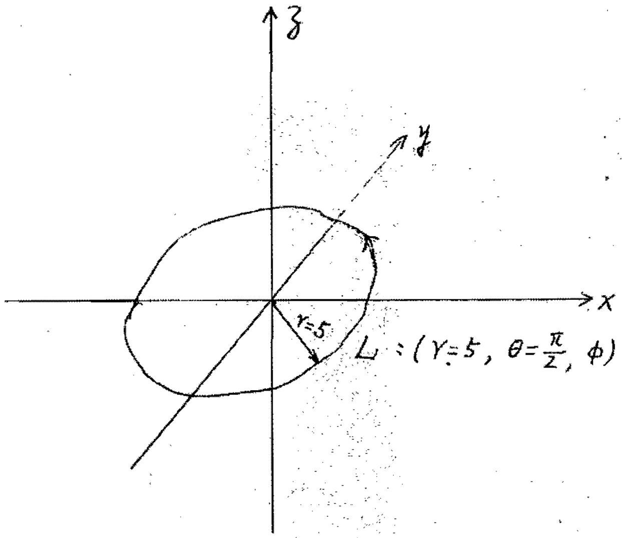
(ii) L : a parabolic path described by $y = x^2$ from $P_1$ to $P_2$ on $z = 3$ plane . Compare this answer with that obtained in (b)(i) and comment on whether the given $\vec{F}$ is a conservative vector field or not. **( 7 marks )**

(iii) Find $\vec{\nabla} \times \vec{F}$ . Does this result agree with the comment you made in (b)(ii)? **( 5 marks )**

## Question two

Given $\vec{F} = \vec{e}_r \left(r^2\right) + \vec{e}_\theta \left(3r^2 \sin(\phi)\right) + \vec{e}_z \left(6\, r^2 \sin(\theta)\right)$ in spherical coordinates,

(a) find the value of $\oint_L \vec{F} \bullet d\vec{l}$ if $L$ is the circular closed loop of radius 5 on $\theta = \dfrac{\pi}{2}$ *plane* in counter clockwise sense as shown in the diagram below




i.e.,

$$L : \left( r = 5 \;,\; \theta = \frac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{l} = +\vec{e}_\phi\, r \sin(\theta)\, d\phi \xrightarrow{r=5 \;\&\; \theta=\frac{\pi}{2}} \vec{e}_\phi\, 5\, d\phi \right)$$

**( 8 marks )**

(b) (i) find $\vec{\nabla} \times \vec{F}$ , **( 7 marks )**

(ii) then evaluate the value of $\iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ where S is bounded by L given in (a) , i.e.,

$$S : \left( \begin{array}{l} 0 \le r \le 5 \;,\; \theta = \dfrac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \quad \& \quad d\vec{s} = -\vec{e}_\theta\, r \sin(\theta)\, dr\, d\phi \\ \qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\theta=\frac{\pi}{2}} -\vec{e}_\theta\, r\, dr\, d\phi \end{array} \right)$$

Compare this value with that obtained in (a) and make a brief comment.

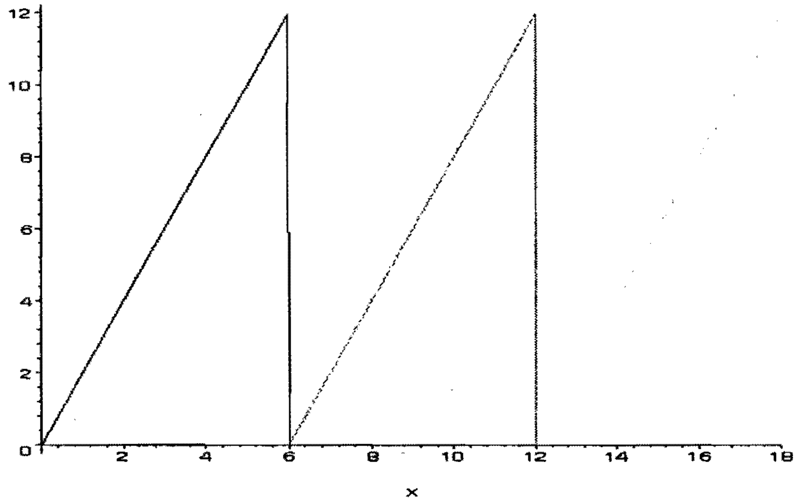
**( 10 marks )**

## Question three

Given the following non-homogeneous differential equation as $\dfrac{d^2 x(t)}{dt^2} + 2\dfrac{d\,x(t)}{dt} + 5\,x(t) = f(t)$ ,

where $f(t)$ is a periodic jigsaw shape driving force of period 6 , i.e.,

$f(t) = f(t+6) = f(t+12) = \cdots\cdots$ and plotted against its first three periods as shown below :




i.e., its first period description is $f(t) = 2\,t \quad for \quad 0 \le t \le 6$

(a) express $f(t)$ in terms its Fourier series , i.e.,

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{3}\right) + b_n \sin\left(\frac{n\pi t}{3}\right)\right)$$ , and show that

$a_0 = 6 \quad , \quad a_n = 0 \;\; for \;\; n = 1,2,3,\cdots \;\; \& \;\; b_n = -\dfrac{12}{n\pi} \;\; for \;\; n = 1,2,3,\cdots$ **( 10 marks )**

$$\left(\begin{array}{l} Note\ :\ a_0 = \dfrac{1}{6}\displaystyle\int_{t=0}^{6} f(t)\,dt \quad , \quad a_n = \dfrac{1}{3}\displaystyle\int_{t=0}^{6} f(t)\cos\left(\dfrac{n\pi t}{3}\right)dt \quad and \\ b_n = \dfrac{1}{3}\displaystyle\int_{t=0}^{6} f(t)\sin\left(\dfrac{n\pi t}{3}\right)dt \quad for \quad n = 1,2,3,\cdots\cdots \end{array}\right)$$

(b) find its particular solution $x_p(t)$ corresponding to

(i) $f(t) = 6$ , and named as $x_p^{(0)}(t)$ , **( 3 marks )**

(ii) $f(t) = -\dfrac{12}{n\pi}\sin\left(\dfrac{n\pi t}{3}\right)$ , and named as $x_p^{(n)}(t)$ , in terms of $n$ . **( 12 marks )**

## Question four

(a) Given the following 2-D Laplace equation in cylindrical coordinates as

$$\nabla^2 f(\rho,\phi)=0=\frac{\partial^2 f(\rho,\phi)}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f(\rho,\phi)}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f(\rho,\phi)}{\partial\phi^2} \quad \text{, set } f(\rho,\phi)=F(\rho)\,G(\phi)$$

and use separation variable scheme to separate the above partial differential equation into two ordinary differential equations. **( 5 marks )**

(b) Given a Legendre's differential equation as :

$$\left(1-x^2\right)\frac{d^2 y(x)}{d x^2}-2x\frac{d y(x)}{d x}+20\,y(x)=0 \text{ , set } y(x)=\sum_{n=0}^{\infty} a_n x^{n+s} \quad \& \quad a_0\neq 0 \text{ and}$$

utilize the power series method,

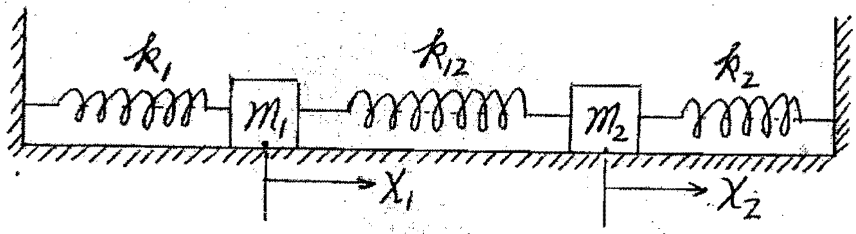
(i) write down its indicial equations and show that $s=0$ *or* $1$ and $a_1=0$ ,

**( 8 marks )**

(ii) write down its recurrence relation. Set $a_0=1$ and use the recurrence relation to generate two independent solutions in power series form truncated up to $a_6$ term. Show that one of the independent solution is a polynomial. **( 12 marks )**

**Question five**

Two simple harmonic oscillators are joined by a spring with a spring constant $k_{12}$ as shown in the diagram below :




The equations of motion for this coupled oscillator system ignoring friction are given as

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -(k_1 + k_{12})\, x_1(t) + k_{12}\, x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = k_{12}\, x_1(t) - (k_2 + k_{12})\, x_2(t) \end{cases}$$

where $x_1$ & $x_2$ are horizontal displacements of $m_1$ & $m_2$ measured from their respective resting positions.

If given $m_1 = 1\ kg$ , $m_2 = 2\ kg$ , $k_1 = 4\ \dfrac{N}{m}$ , $k_2 = 8\ \dfrac{N}{m}$ & $k_{12} = 6\ \dfrac{N}{m}$ ,

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$ , then the above given equations can be deduced to the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -10 & 6 \\ 3 & -7 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **( 5 marks )**

(b) find the eigenfrequencies $\omega$ of the given coupled system , **( 6 marks )**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b), **( 6 marks )**

(d) find the normal coordinates of the given coupled system , **( 6 marks )**

(e) write down the general solutions for $x_1(t)$ & $x_2(t)$ . **( 2 marks )**

## Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(F_1 h_2 h_3)}{\partial u_1} + \frac{\partial(F_2 h_1 h_3)}{\partial u_2} + \frac{\partial(F_3 h_1 h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1 F_1 + \vec{e}_2 F_2 + \vec{e}_3 F_3$ and

| | | | |
|---|---|---|---|
| $(u_1, u_2, u_3)$ | represents | $(x, y, z)$ | for rectangular coordinate system |
| | represents | $(\rho, \phi, z)$ | for cylindrical coordinate system |
| | represents | $(r, \theta, \phi)$ | for spherical coordinate system |
| $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ | represents | $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ | for rectangular coordinate system |
| | represents | $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$ | for cylindrical coordinate system |
| | represents | $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ | for spherical coordinate system |
| $(h_1, h_2, h_3)$ | represents | $(1, 1, 1)$ | for rectangular coordinate system |
| | represents | $(1, \rho, 1)$ | for cylindrical coordinate system |
| | represents | $(1, r, r\sin(\theta))$ | for spherical coordinate system |

$$\int (t\sin(kt))\,dt = -\frac{t\cos(kt)}{k} + \frac{\sin(kt)}{k^2}$$

$$\int (t\cos(kt))\,dt = \frac{t\sin(kt)}{k} + \frac{\cos(kt)}{k^2}$$