UNIVERSITY OF SWAZILAND

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FACULTY OF SCIENCE

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2011/2012

TITLE OF PAPER	:	CLASSICAL MECHANICS
COURSE NUMBER	:	P320
TIME ALLOWED	:	THREE HOURS
INSTRUCTIONS	:	ANSWER ANY FOUR OUT OF FIVE QUESTIONS. EACH QUESTION CARRIES 25 MARKS. MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

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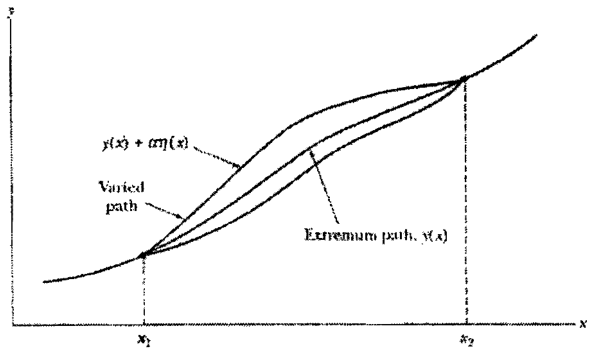
P320 CLASSICAL MECHANICS

Question one

(a) Given the following definite integral of $J(\alpha)=\int_{x_1}^{x_2} f(y(\alpha,x),y'(\alpha,x),y''(\alpha,x);x)\,dx$, where the varied integration path is $y(\alpha,x)=y(x)+\alpha\,\eta(x)$, $\eta(x_1)=\eta(x_2)=0$ *and* $\left.\frac{d\eta(x)}{dx}\right|_{x=x_1}=\left.\frac{d\eta(x)}{dx}\right|_{x=x_2}=0$ as shown in the following diagram :



Using the extremum condition for $J(\alpha)$, i.e., $\left.\frac{\partial J(\alpha)}{\partial\alpha}\right|_{\alpha=0}=0$, to deduce that

f *along the extremum path, i.e.,* $f(y(x),y'(x),y''(x);x)$, satisfies the following equation:

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0 \ .$$ **(13 marks)**

(b) For a certain dynamical system the kinetic energy T and potential energy V are given by

$$T=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_1\,\dot{q}_2+\dot{q}_2^2\right)$$

$$V=\frac{3}{2}q_2^2$$

where q_1 , q_2 are the generalized coordinates.

(i) Write down Lagrange's equations of motion. **(6 marks)**

(ii) Deduce an expression for q_2 in terms of t . **(6 marks)**

Question two

(i) Show that for two dimensional motion of a particle of mass m with a constant acceleration α along $+x$ direction and a zero acceleration along y direction the Hamiltonian is given by the expression,

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - m\,\alpha\,x$$

(6 marks)

(ii) From the definition of the Poisson brackets , i.e., $[u,v] \equiv \sum_{\alpha}\left(\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right)$, evaluate $[x,H]$ *and* $[p_x,H]$.

(8 marks)

(iii) For an equation of the type $\frac{du}{dt} = [u,H]$ the specific solution of $u(t)$ is given by the following series expansion

$$u(t) = u_0 + [u,H]_0\, t + [[u,H],H]_0 \frac{t^2}{2!} + [[[u,H],H],H]_0 \frac{t^3}{3!} + \cdots\cdots$$

where subscript 0 denotes the initial conditions at $t = 0$.
Use the above relation to show that for the given Hamiltonian, the specific solution of $x(t)$ is given by

$$x(t) = x_0 + \frac{p_{x,0}}{m} t + \frac{\alpha}{2} t^2$$

where x_0 *and* $p_{x,0}$ are the initial x-position and x-momentum respectively.

(11 marks)

Question three

(a) Given the Lagrangian for the two-body central force system as :

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

where μ is the reduced mass of the system and (r, θ) are polar coordinates of the motion plane with its origin at the center of mass of the two-body system.

(i) Write down the Lagrange's equation for θ and show that the angular momentum l is conserved , i.e., deduce that

$$\dot{\theta} = \frac{l}{\mu r^2} \quad \cdots\cdots (1)$$

where l is a constant. **(3 marks)**

(ii) Write down the Lagrange's equation for r , with eq.(1) inserted, deduce that

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} + \frac{dU(r)}{dr} = 0 \quad \cdots\cdots (2)$$

(3 marks)

(iii) Multiply eq.(2) by dr and use $\ddot{r}\,dr = \frac{d\dot{r}}{dt}dr = d\dot{r}\frac{dr}{dt} = \dot{r}\,d\dot{r} = d\left(\frac{\dot{r}^2}{2}\right)$ to show that the total energy $E\,(\equiv T + U)$ is conserved . **(6 marks)**

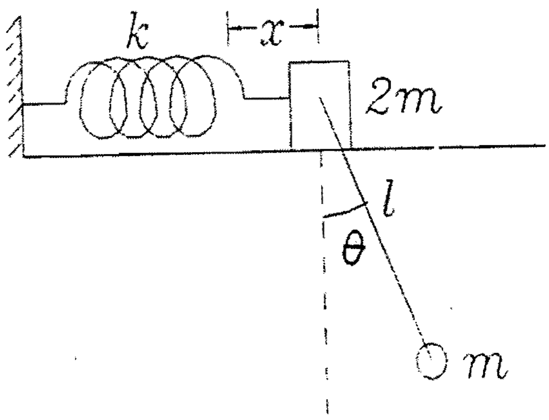
(b) If an earth satellite of 100 kg mass is having a pure tangential speed $v_\theta\left(= r\dot{\theta}\right) = 10{,}000\ \frac{m}{s}$ at its near-earth-point 600 km above the earth surface,

(i) calculate the values of l *and* E of this satellite , **(6 marks)**

(ii) calculate the values of the eccentricity ε and show that the orbit is an elliptical orbit. Also calculate its period. **(7 marks)**

Question four

A simple pendulum of mass m and length l, hangs from a supporting block of mass $2m$ which can move along a horizontal line (in the plane of the pendulum), and is restricted by a spring with a spring constant k as shown below



(i) For small θ, i.e., $\left(\sin(\theta) \approx \theta \ and \ \cos(\theta) \approx 1 - \frac{\theta^2}{2}\right)$, show that the Lagrangian for the system can be expressed as:

$$L = \frac{3}{2} m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + m l \dot{x} \dot{\theta} - \frac{1}{2} m g l \theta^2 - \frac{1}{2} k x^2$$

where the zero gravitational potential is set at the equilibrium position. **(6 marks)**

(ii) Write down the equations of motion and deduce that

$$\begin{cases} \ddot{x} = -\left(\frac{k}{2m}\right) x + \left(\frac{g}{2}\right) \theta \\ \ddot{\theta} = \left(\frac{k}{2ml}\right) x - \left(\frac{3g}{2l}\right) \theta \end{cases}$$

(10 marks)

(iii) Set $x = \hat{X}_1 e^{i\omega t}$ and $\theta = \hat{X}_2 e^{i\omega t}$ (where $\hat{X}_1$ and $\hat{X}_2$ are constants) and deduce from the equations in (ii) the matrix equation $-\omega^2 X = A X$ where

$$X = \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} \quad and \quad A = \begin{pmatrix} -\left(\frac{k}{2m}\right) & \left(\frac{g}{2}\right) \\ \frac{k}{2ml} & -\left(\frac{3g}{2l}\right) \end{pmatrix}$$

(4 marks)

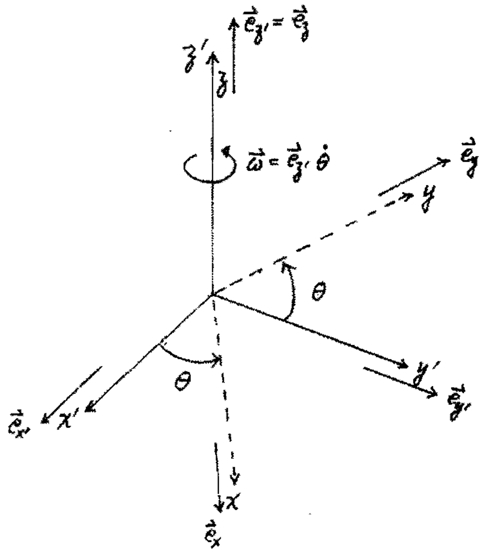
(iv) Show that the eigenfrequencies ω of this coupled system satisfies the following equation

$$\omega^4 - \left(\frac{k}{2m} + \frac{3g}{2l}\right) \omega^2 + \left(\frac{kg}{2ml}\right) = 0$$

(5 marks)

Question five

(a) Two set of Cartesian coordinate axes are having the same origins and z-axis. The non-prime system (referred to as "rotating" system) is rotating with an angular velocity $\vec{\omega} = \vec{e}_{z'}\,\dot{\theta}$ about the prime system (referred as "fixed" system) as shown below:



For any vector field $\vec{F}$ decomposed into the above two-set of cartesian components, i.e., $\vec{F} = \vec{e}_x\,F_x + \vec{e}_y\,F_y + \vec{e}_z\,F_z = \vec{e}_{x'}\,F_{x'} + \vec{e}_{y'}\,F_{y'} + \vec{e}_{z'}\,F_{z'}$, show that

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \left(\frac{d\vec{F}}{dt}\right)_{rotating} + \vec{\omega}\times\vec{F}$$ where

$$\left(\frac{d\vec{F}}{dt}\right)_{fixed} = \vec{e}_{x'}\,\frac{dF_{x'}}{dt} + \vec{e}_{y'}\,\frac{dF_{y'}}{dt} + \vec{e}_{z'}\,\frac{dF_{z'}}{dt} \quad and$$

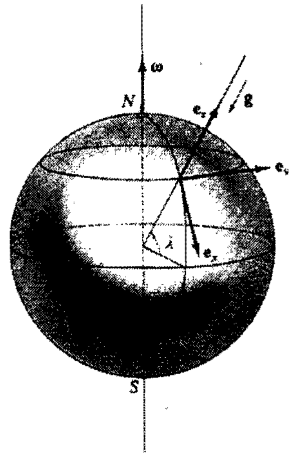
(12 marks)

$$\left(\frac{d\vec{F}}{dt}\right)_{rotating} = \vec{e}_x\,\frac{dF_x}{dt} + \vec{e}_y\,\frac{dF_y}{dt} + \vec{e}_z\,\frac{dF_z}{dt}$$

(Hint : $\vec{e}_x = \vec{e}_{x'}\cos(\theta) + \vec{e}_{y'}\sin(\theta)$, $\vec{e}_y = -\vec{e}_{x'}\sin(\theta) + \vec{e}_{y'}\cos(\theta)$ *and* $\vec{e}_z = \vec{e}_{z'}$)

Question five (continued)

(b)



If a particle is projected vertically upward with an initial speed v_0 to a height h above a point on the earth's surface at northern latitude λ , show that it strikes the ground at a point $\frac{4}{3}\,\omega\cos(\lambda)\sqrt{\frac{8\,h^3}{g}}$ to the west. Neglect air resistance and only consider small vertical height. **(13 marks)**

(Hint :

$\vec{a}_{eff} \approx \vec{e}_z\,(-g) - 2\,\vec{\omega}\times\vec{v}_r$, $\vec{v}_r \approx \vec{e}_z\,(v_0 - g\,t)$, $\vec{\omega} = \vec{e}_x\,(-\omega\cos(\lambda)) + \vec{e}_z\,(\omega\sin(\lambda))$

and $v_0 = \sqrt{2\,g\,h}$, $(\textit{total time for the given motion}) = \frac{2\,v_0}{g}$)

Useful informations

$$V = -\int \vec{F} \bullet d\vec{l} \quad and \quad d\vec{l} = \vec{e}_x\, x + \vec{e}_y\, y + \vec{e}_z\, z \quad in\, cartesian$$

$$L = T - V = L(q_\alpha , \dot{q}_\alpha ; t)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$$

$$H = \sum_\alpha p_\alpha\, \dot{q}_\alpha - L = H(q_\alpha , p_\alpha ; t) \xrightarrow{if\ \frac{\partial H}{\partial t} = 0} T + V$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$$[u , v] \equiv \sum_\alpha \left(\frac{\partial u}{\partial q_\alpha} \frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha} \frac{\partial v}{\partial q_\alpha} \right)$$

$$G = 6.673 \times 10^{-11}\ \frac{N\, m^2}{kg^2}$$

$$radius\ of\ earth \quad r_E = 6.4 \times 10^6\ \ m$$

$$mass\ of\ earth \quad m_E = 6 \times 10^{24}\ \ kg$$

$$earth\ attractive\ potential \equiv -\frac{k}{r} \quad where \quad k = G\, m\, m_E$$

$$\varepsilon = \sqrt{1 + \frac{2\, E\, l^2}{\mu\, k^2}} \quad \{(\varepsilon = 0 , circle), (0 < \varepsilon < 1 , ellipse), (\varepsilon = 1 , parabola), \cdots\}$$

$$\mu = \frac{m_1\, m_2}{m_1 + m_2} \approx m_1 \quad if \quad m_2 >> m_1$$

$$For\ elliptical\ orbit, i.e., 0 < \varepsilon < 1 , then \begin{cases} semi-major \ \ a = \dfrac{k}{2\,|E|} \\ semi-minor \ \ b = \dfrac{l}{\sqrt{2\, \mu\, |E|}} \\ period \ \ \tau = \dfrac{2\,\mu}{l} (\pi\, a\, b) \end{cases}$$

$$I = \begin{pmatrix} \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,2} & -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,1} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,3}^2\right) & -\sum_\alpha m_\alpha\, x_{\alpha,2}\, x_{\alpha,3} \\ -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,1} & -\sum_\alpha m_\alpha\, x_{\alpha,3}\, x_{\alpha,2} & \sum_\alpha m_\alpha \left(x_{\alpha,1}^2 + x_{\alpha,2}^2\right) \end{pmatrix}$$

$$\vec{F}_{eff} = \vec{F} - m\,\ddot{\vec{R}}_f - m\,\dot{\vec{\omega}} \times \vec{r} - m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\,m\,\vec{\omega} \times \vec{v}_r \qquad \textit{where}$$

$\vec{r}\,' = \vec{R} + \vec{r}$ *and*

$\vec{r}\,'$ *refers to fixed(inertial system)*

$\vec{r}$ *refers to rotatinal(non – inertial system) rotates with* $\vec{\omega}$ *to* $\vec{r}\,'$ *system*

$\vec{R}$ *from the origin of* $\vec{r}\,'$ *to the origin of* $\vec{r}$

$$\vec{v}_r = \left(\frac{d\,\vec{r}}{d\,t}\right)_r$$