

UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE

197

DEPARTMENT OF PHYSICS

MAIN EXAMINATION: 2011/2012

TITLE OF THE PAPER: COMPUTATIONAL METHODS-II

COURSE NUMBER: P482

**TIME ALLOWED:**

SECTION A: ONE HOUR

SECTION B: TWO HOURS

**INSTRUCTIONS:**

THE ARE TWO SECTIONS IN THIS PAPER:

- **SECTION A** IS A WRITTEN PART. ANSWER THIS SECTION ON THE ANSWER BOOK. IT CARRIES A TOTAL OF **30** MARKS.
- **SECTION B** IS A PRACTICAL PART WHICH YOU WILL WORK ON A PC AND SUBMIT THE PRINTED OUTPUT. IT CARRIES A TOTAL OF **70** MARKS.

Answer any two questions from **section A** and all the questions from **section B**.

Marks for different sections of each Question are shown in the right hand margin.

THE PAPER HAS 7 PAGES, INCLUDING THIS PAGE.

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## Section A

### Question 1

The heat equation that describes the flow of heat from regions of high temperature to regions of low temperature in one dimension is given as

$$\frac{\partial T(x, t)}{\partial t} = D \left( \frac{\partial^2 T(x, t)}{\partial x^2} \right)$$

where  $D$  is the diffusivity.

- (a) Using the forward difference for the time derivative and the usual finite difference for the spatial derivative, write the discretized version of the heat equation that can be used to numerically solve for  $T[i, n + 1] = T[i\Delta x, (n + 1)\Delta t]$ , where  $\Delta x$  is the grid spacing and  $\Delta t$  is the time step.

[5 marks]

- (b) Write a Maple script to calculate the temperature profile of a 10 m rod at 100 equally spaced points. The profile is to be calculated at a time step of  $\Delta t = 0.2$ s for a total time of 20 seconds. Assume that  $D = 0.4\text{m}^2/\text{s}$ , the initial temperature of the rod is

$$T(x, t = 0) = 100K,$$

and the rod is insulated along its length with its ends in contact with ice i.e.,

$$T(x = 0, t) = T(x = 10\text{m}, t) = 273K.$$

[10 marks]

## Question 2

- (a) One of the simplest random number generator has its sequence of numbers given by

$$x_{i+1} = (ax_i + b) \bmod c$$

where  $a$ ,  $b$ , and  $c$  are magic numbers.

- (i) State three most important criteria for a good uniform random number generator.

[3 marks]

- (ii) One option of random numbers could be:  $a = 7^5$ ,  $b = 0$ , and  $c = 2^{31} - 1$ . What is the period of the number generator in this case?

[1 mark]

- (b) What is the range of the sequence generated by the built-in uniform random number generator in Maple.

[1 mark]

- (c) In the two dimensional Ising model for magnetic systems, the magnetic spin at site  $(i, j)$  is given by  $S[i, j] = \pm 1$ . The plus represents a spin-up and the minus represents a spin-down. In the paramagnetic state (non-magnetic state), the spins' orientation at the lattice sites is random. Write a Maple script that generates the paramagnetic state for the Ising model on a grid with  $128 \times 128$  lattice points.

[10 marks]

### Question 3

(a) A continuous voltage signal  $V(t)$  is sampled at a rate

$$f_s = \frac{1}{\Delta t}$$

for  $N$  equally spaced points.

- (i) What is the period  $T$  of the measurement.
- (ii) What is the lowest nonzero frequency in the Fourier representation (or power spectrum) of  $V(t)$ .
- (iii) According to the Sampling Theorem, what is the highest frequency in the Fourier representation of  $V(t)$ .
- (iv) Sketch the power spectrum of white noise.
- (v) Sketch the power spectrum of the continuous signal  $H(t) = 1.0 + \sin(t)$

[5 marks]

(b) The following discretized signal

$$g(t_i) = \cos(3i\Delta t) + \sin(i\Delta t/2).$$

is sampled at time interval  $\Delta t = \pi/3$  for  $N = 256$  equally spaced points. Using some analytical arguments show that the algorithm

$$w_i = (g_i + g_{i+1})/2$$

acts as a low pass filter for the signal  $g_i$ .

[5 marks]

(c) Write a Maple script that constructs the signal

$$y(t_i) = \cos(5t_i) + r_i$$

where  $t_i = i \cdot \Delta t$  for  $i = 1..100$  and the additive noise,  $-0.1 \leq r_i \leq 0.1$ , is obtained from using the Maple's random number generator. Sketch the power spectrum of the signal.

[5 marks]

## Section B

### Question 4

Given the continuous signal

$$d(t) = 3 \sin(2\pi f_0 t) + 2 \sin(2\pi f_1 t) + \cos(2\pi f_2 t)$$

where  $f_0 = 28\text{Hz}$ ,  $f_1 = 84\text{Hz}$ , and  $f_2 = 140\text{Hz}$ , state the appropriate limit for the sample rate  $f_s = 1/\Delta t$  such that the signal that can be reconstructed from the Fourier transform of  $d(t)$  is consistent with it. With your appropriately chosen value of  $\Delta t$  sample the above signal for a suitable period  $T = N\Delta t$ , where  $N$  is an integer.

- (a) Plot the sampled signal for  $t = 0$  to  $t = N\Delta t/2$ .
- (b) Compute the Fourier Transform of the signal,  $D(\omega)$ , and plot the power spectrum.
- (c) Reconstruct the time dependent signal  $d(t)$  from  $D(\omega)$  by utilizing the inverse Fourier Transform procedure *iFFT*. Compare the reconstructed signal with the original signal. Discuss your results.

[15 marks]

## Question 5

A realistic projectile motion includes the effects of the air resistance. The force due acting upon the projectile due to air drag is

$$\mathbf{F}_{drag} = -B_2 v^2 \hat{\mathbf{v}},$$

where  $B_2$  is the drag constant and  $\mathbf{v} = (v_x, v_y)$  is the velocity of the projectile. Newton's second law for a realistic projectile is then given as

$$m\mathbf{r}'' = \mathbf{F}_g + \mathbf{F}_{drag},$$

where  $\mathbf{F}_g$  is the contribution of the gravitational force. This equation can be decomposed into a set of two equations of motion:

$$\begin{aligned} m \frac{d^2 x(t)}{dt^2} &= -B_2 v_x(t) \sqrt{v_x^2(t) + v_y^2(t)}, \\ m \frac{d^2 y(t)}{dt^2} &= -mg - B_2 v_y(t) \sqrt{v_x^2(t) + v_y^2(t)}. \end{aligned}$$

The above two equations can be represented by four discretized equations

$$\begin{aligned} x[i+1] &= x[i] - \Delta t v_x[i] \\ y[i+1] &= y[i] - \Delta t v_y[i] \\ v_x[i+1] &= v_x[i] - \Delta t (B_2 v_x[i] \sqrt{v_x^2[i] + v_y^2[i]}) / m \\ v_y[i+1] &= v_y[i] - \Delta t (B_2 v_y[i] \sqrt{v_x^2[i] + v_y^2[i]}) / m \end{aligned}$$

where  $x[i] = x(t_i)$  and  $t_i = \Delta t$ . Fig. 1 shows a simulation of a spherical ball with a radius  $r = 5$  cm in air. It is launched from the origin with an angle  $45^\circ$  with respect to the horizontal and an initial speed of  $700 \text{ ms}^{-1}$ . A timestep of  $0.05\text{s}$  was employed in the numerical integration. The figure on the left shows a graph of  $y(t)$  against  $x(t)$  and the figure on the right shows the graph of  $v_y(t)$  vs  $v_x(t)$ . The drag coefficient is given as  $B_2/m = 0.005\text{m}^{-1}$ .

(a) Write a program in Maple using the above discretized equations of motion to reproduce the results shown in Fig. 1. Discuss the role played by the drag force.

[20 marks]

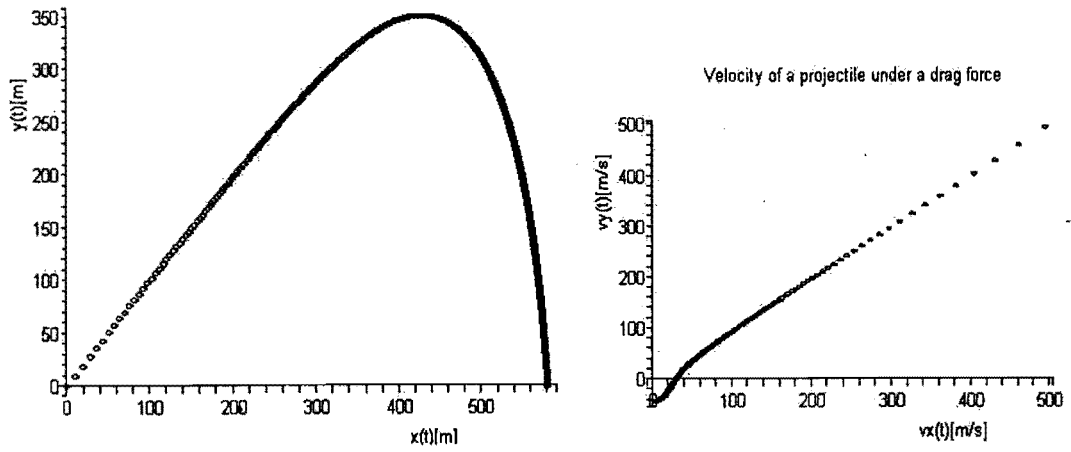


FIG. 1: Trajectory and velocities of a realistic projectile.

- (b) In warfare you generally want to hit a particular target (as opposed to having the shells land indiscriminately). However, this is not easy, since very small changes in any parameter can lead to large changes in the range. Investigate the effects of small changes in the initial conditions to the trajectory of the projectile. Discuss your results

[15 marks]

- (c) A more realistic projectile will also include the effects of reduced air density at high altitudes. The density of the atmosphere varies as

$$\rho(y) = \rho_0 \exp(-y/y_0),$$

where  $y$  is the altitude,  $y_0 = 1.0 \times 10^3$  m. The drag force is then modified to

$$\tilde{\mathbf{F}}_{drag} = \frac{\rho}{\rho_0} \mathbf{F}_{drag}$$

Calculate the trajectory of the projectile including the effects of the reduced air density at high altitudes. Perform your calculation for different firing angles and determine the value of the angle that gives the maximum range. Let the drag coefficient  $B_2/m = 4 \times 10^{-5} \text{ m}^{-1}$ .

[20 marks]

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