UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2012/2013

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| TITLE OF PAPER | : | ELECTRICITY AND MAGNETISM |
| COURSE NUMBER | : | P221 |
| TIME ALLOWED | : | THREE HOURS |
| INSTRUCTIONS | : | ANSWER ANY FOUR OUT OF FIVE QUESTIONS.<br>EACH QUESTION CARRIES 25 MARKS.<br>MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. |

THIS PAPER HAS ELEVEN PAGES, INCLUDING THIS PAGE.

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## P221 ELECTRICITY AND MAGNETISM

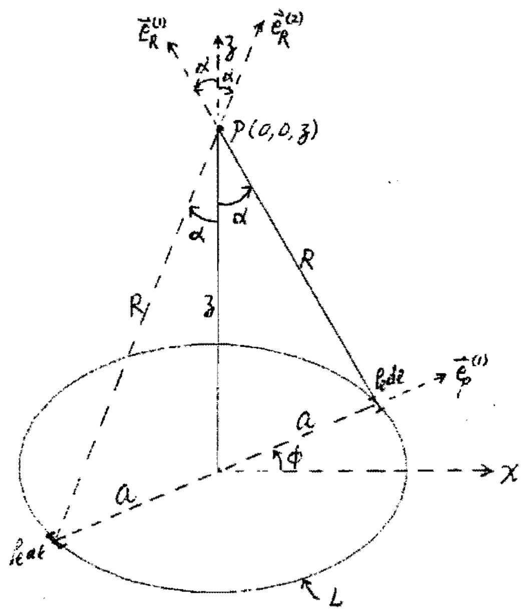
**Question one**

(a) The integral form of Faraday's law is $\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_s \vec{B} \bullet d\vec{s}\right)$.

(i) Explain briefly the meaning of both sides of the above equation. **( 3 marks )**

(ii) Which physical law is also built into the above equation? **( 1 marks )**

(b) An uniformly charged circular ring $L$ with constant line charge density $\rho_l$ is having a radius of $a$ and situated on $z = 0$ plane with the ring's centre at the origin as shown in the following diagram



(i) Find the total electric field $\vec{E}$ at the field point P (x = 0 , y = 0 , z) due to the given charged circular ring by using $\vec{E} = \oint_L \vec{e}_R \frac{\rho_l \, dl}{4\pi\varepsilon_0 R^2}$ and deduce that

$$\vec{E} = \vec{e}_z \frac{\rho_l \, a \, z}{2\varepsilon_0 R^3} \quad where \quad R = \sqrt{z^2 + a^2}$$ **( 10 marks )**

(Hint : use "pair" addition and then integrate for half of the ring.)

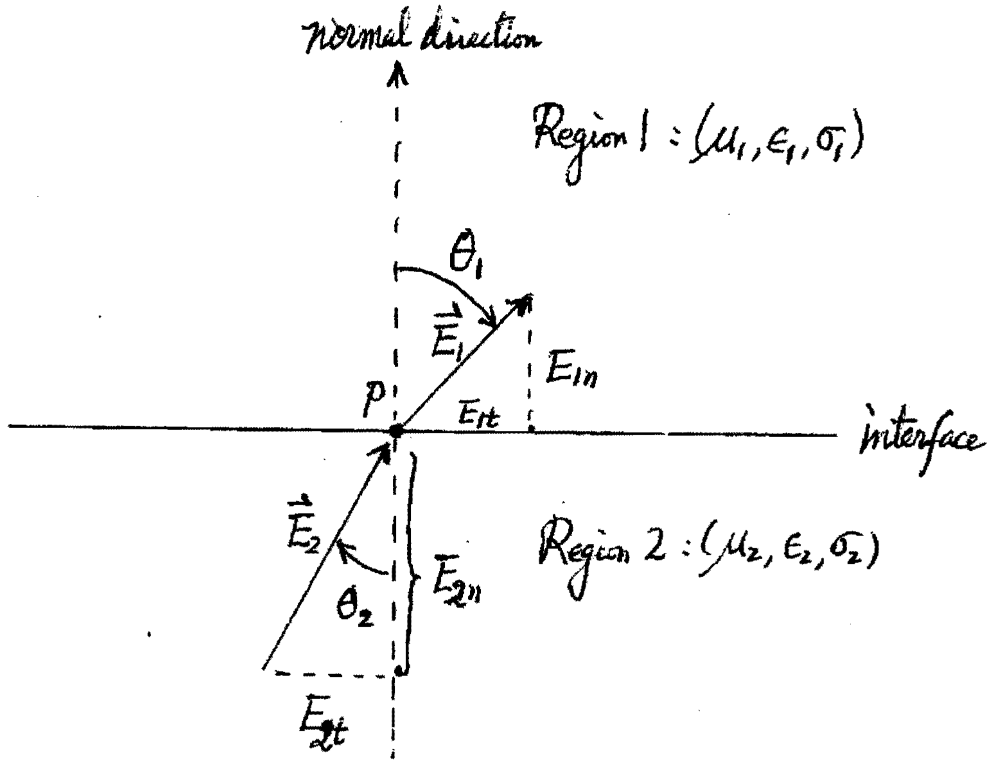
(ii) Find the electric potential $V$ at the field point P (x = 0 , y = 0 , z) due to the given charged circular ring by using $V = \oint_L \frac{\rho_l \, dl}{4\pi\varepsilon_0 R}$ and deduce that

$$V = \frac{\rho_l \, a}{2\varepsilon_0 R}$$ **( 4 marks )**

(iii) Use above $V$ to find the electric field $\vec{E}$ at the field point P and show that it's the same as that obtained in (b)(i). **( 7 marks )**

**Question two**

(a) For any point P on the interface separating two material regions characterized by their respective parameters of $(\mu_1, \varepsilon_1, \sigma_1)$ & $(\mu_2, \varepsilon_2, \sigma_2)$ as shown in the diagram below

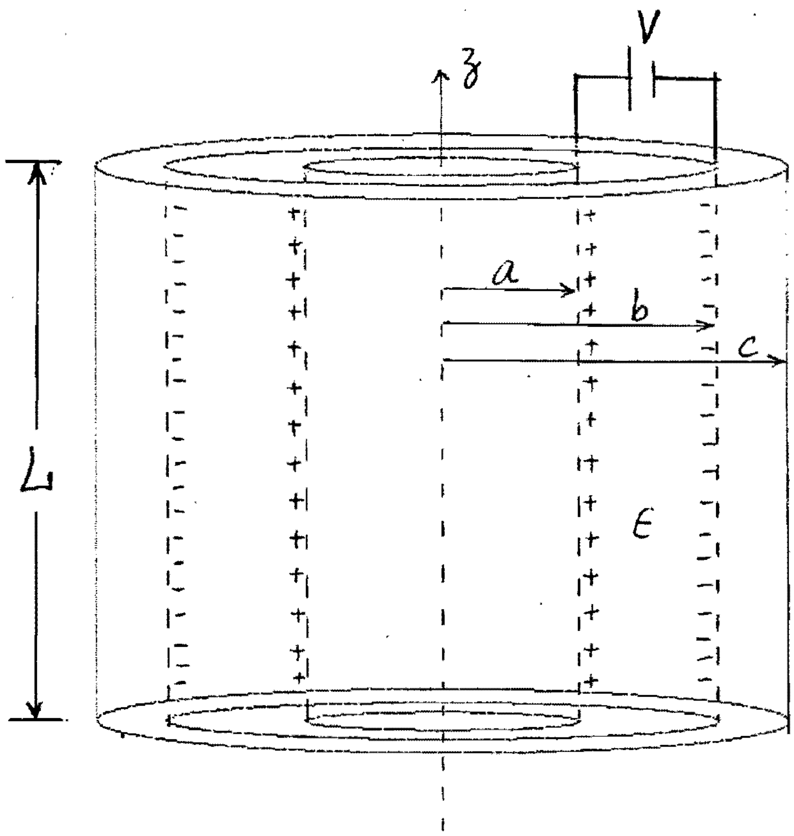



where $\vec{E}_1$ & $\vec{E}_2$ are the electric field right above and below the point P. Use the boundary conditions that both the tangential component of $\vec{E}$ and the normal component of $\vec{D}$ are continuous at the interface to deduce the following refractive law for $\vec{E}$ as

$$\frac{\tan(\theta_1)}{\tan(\theta_2)} = \frac{\varepsilon_1}{\varepsilon_2}$$ **( 5 marks )**

(b) A coaxial cable with its central axis coinciding with the z-axis and its total length L is shown in the following diagram

## Question two (continued)




where $a$ is the radius of the inner cable and $c-b$ is the thickness of the outer hollow cable. In-between the cables is filled with the insulating material of permittivity $\varepsilon$.

(i) Find the surface charge densities on both $\rho = a$ & $\rho = b$ conductor's surfaces and then use the boundary conditions on conductor's surface to find $\vec{D}$ & $\vec{E}$ on both $\rho = a$ & $\rho = b$ conductor's surfaces. **( 7 marks )**

(ii) Use integral electric Gauss law for $\vec{D}$, choose and draw a proper closed surface to find $\vec{D}$ for the $a < \rho < b$ region and then find $\vec{E}$ *for the* $a < \rho < b$ *region*

**( 7 marks )**

(iii) Find the potential difference between the inner and outer conducting cable surfaces V by evaluating the following integral $V = -\int_{\rho=b}^{a} \vec{E} \bullet d\vec{l}$ where

$d\vec{l} = \vec{a}_\rho\, d\rho + \vec{a}_\phi\, \rho\, d\phi + \vec{a}_z\, dz$ . **( 4 marks )**

(iv) Find the distributive capacitance $c\left(\equiv \dfrac{C}{L}\right)$ of this coaxial cable system.

**( 2 marks )**

## Question three

(a) Apply a constant electric field $\vec{E} = \vec{e}_x \, 10^{-5} \, \dfrac{V}{m}$ to a material region with electric permittivity $\varepsilon \, (= 3\, \varepsilon_0)$ and conductivity $\sigma \left( = 2 \times 10^7 \, \dfrac{1}{\Omega\, m} \right)$.

- (i) What would be the value of the resultant polarization field $\vec{P}$ in the given region? **( 4 marks )**
- (ii) What would be the value of the resultant current density $\vec{J}$ in the given region? **( 2 marks )**
- (iii) What would be the value of the average velocity $\vec{v}$ of the conduction electrons in the given region if the number density of the conduction electrons is given as $n = 10^{28} \, \dfrac{conduction\ electrons}{m^3}$. **( 5 marks )**

(b) For any closed surface S in space enclosing a total volume of V ,

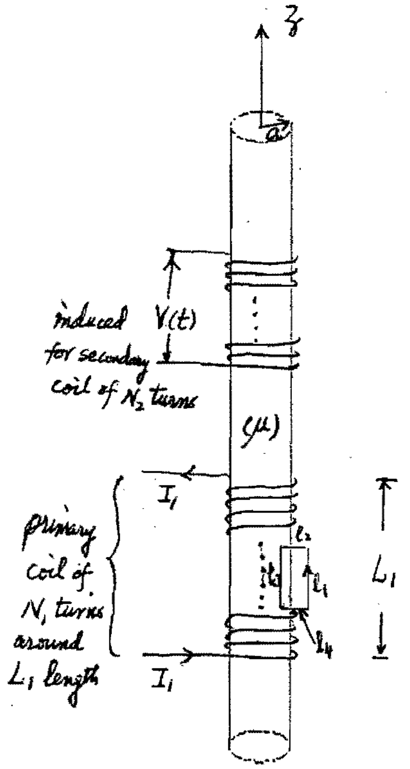
- (i) express the total elctric charge Q in V in terms of the volume charge density $\rho_v$, **( 1 mark )**
- (ii) also express the total current I flowing out of S in terms of the current density $\vec{J}$, **( 1 mark )**
- (iii) and then express the law of conservation of charges based on those integral expressions in (b)(i) and (b)(ii). Then from this integral expression of the law of conservation of charges deduce the differential expression of the same law with the help of the divergence theorem. **( 5 mark )**

(c) Apply a constant magnetic field $\vec{B} = \vec{e}_x \, 10^{-5} \, \dfrac{Wb}{m^2}$ to a material region with magnetic permeability $\mu \, (= 80\, \mu_0)$ .

- (i) What would be the value of the resultant magnetization field $\vec{M}$ in the given region? **( 5 marks )**
- (ii) Describe the value range of $\mu$ comparing to the value of $\mu_0$ for diamagnetic, paramagnetic and ferromagnetic materials respectively and then indicate what type of magnetic material the given region is made of. **( 2 marks )**

## Question four

(a) A static current $I_1$ flows in the $N_1$ turn toroid wired around an iron core of cross section radius $a$ with a permeability $\mu$, with its central axis coincide with the z – axis as shown below




(i) Use the closed loop $(l_1 + l_2 + l_3 + l_4)$ drawn in the given diagram where $\vec{l}_1 = \vec{e}_z\, b\,(outside\ \ the\ core),\ \vec{l}_2 = -\,\vec{e}_\rho\, c\,,\ \vec{l}_3 = -\,\vec{e}_z\, b\,(inside\ \ the\ core)\ \&\ \vec{l}_4 = \vec{e}_\rho\, c$, set $\vec{B} = \vec{e}_z\, B_z(\rho)\quad for\quad \rho \le a\quad \&\quad \vec{B} = 0\quad for\quad \rho > a$ and use the integral Ampere's law to find $\vec{B}$ and show that $\vec{B} = \vec{e}_z \dfrac{\mu N_1}{L_1} I_1 \quad for \quad \rho \le a$ .

**( 7 marks )**

(ii) Assuming the same $\vec{B}$ obtained in (a)(i) is maintained throughout the iron core (which is a good assumption when $\mu >> \mu_0$ ), then find the total magnetic flux $\Psi_m$ passing through the cross-section area $\pi\, a^2$ of the iron core, i.e.,
$\Psi_m = \int_S \vec{B} \bullet d\vec{s} \quad where\ \ d\vec{s} = \vec{e}_z\ \rho\, d\rho\, d\phi\ \ ,\ \ 0 \le \rho \le a\ \ \&\ \ 0 \le \phi \le 2\pi$ , and show that $\Psi_m = \dfrac{\mu\, N_1\, \pi\, a^2}{L_1} I_1$. **( 3 marks )**

## Question four (continued)

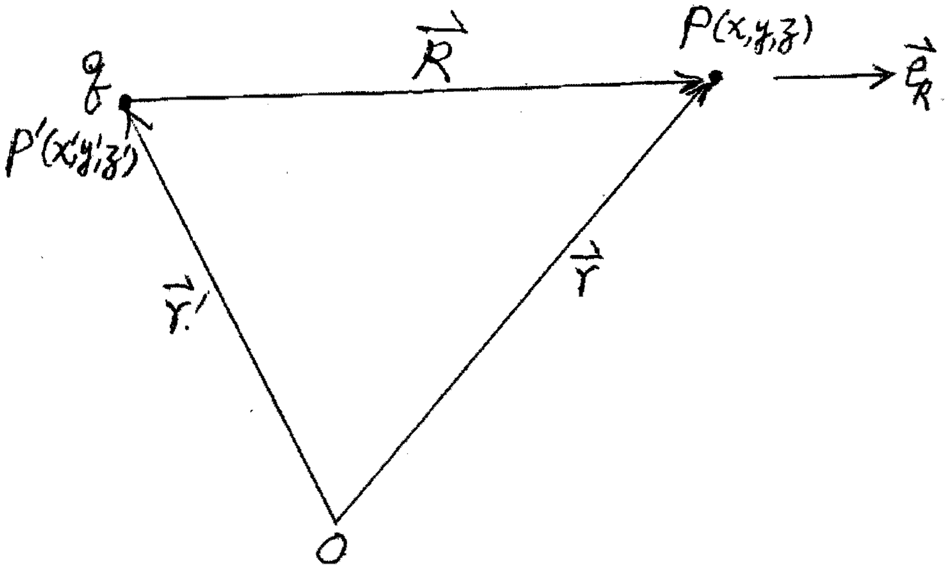
(iii) Find the mutual inductance $M$ between the primary and secondary coils and the self-inductance $L$ of the primary coil in terms of $a\,, L_1\,, N_1\,, N_2\,\&\,\mu$.

**( 6 marks )**

(Hint : The total magnetic flux passing through the primary and secondary coils are $N_1\,\Psi_m$ & $N_2\,\Psi_m$ *respectively where* $\Psi_m$ *is obtained in* (*a*)(*ii*) )

(iv) Find the induced e.m.f. $V(t)$ of the secondary coil in terms of $a\,, L_1\,, N_1\,, N_2\,, \mu\,, I_0\;\&\;\omega$ if the primary coil carries a sinusoidal current of $I_0\sin(\omega t)$ instead of carrying a static current $I_1$ . **( 3 marks )**

(b)




Given that the electric potential V at point P due to a point charge q at point P' as $V(x,y,z)=\dfrac{q}{4\,\pi\,\varepsilon_0\,R}$ *with* $R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ , from $\vec{E}=-\vec{\nabla}V$ and through direct differentiation deduce that

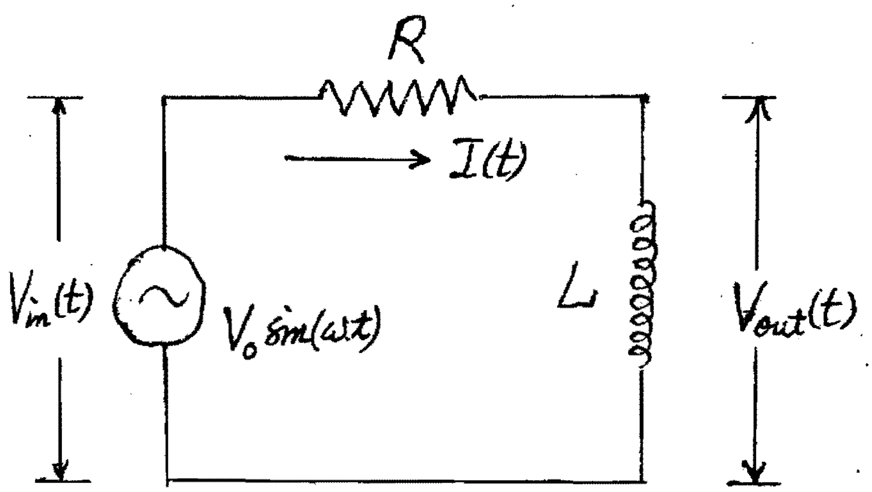
$$\vec{E}=\vec{e}_R\,\frac{q}{4\,\pi\,\varepsilon_0\,R^2}$$

*where* $\vec{R}=\vec{e}_R\,R=\vec{r}-\vec{r}'$ & $\vec{e}_R=\vec{e}_x\,\dfrac{(x-x')}{R}+\vec{e}_y\,\dfrac{(y-y')}{R}+\vec{e}_z\,\dfrac{(z-z')}{R}$

**( 6 marks )**

**Question five**

Connect an AC voltage source $V_{in}(t) = V_0 \sin(\omega t)$ to a RL circuit in series as shown below:




The circuit equation is $V_{in}(t) = R \times I(t) + L \times \dfrac{dI(t)}{dt}$ ...... (1) .

(a) Set its steady state solution of $I(t)$ as $I(t) = k_1 \cos(\omega t) + k_2 \sin(\omega t)$ and substitute it into eq.(1).

(i) Deduce that

$$k_1 = -\frac{V_0 \, \omega \, L}{(\omega \, L)^2 + R^2} \quad \& \quad k_2 = \frac{V_0 \, R}{(\omega \, L)^2 + R^2} \quad \textit{and thus}$$

$$I(t) = -\frac{V_0 \, \omega \, L}{(\omega \, L)^2 + R^2} \cos(\omega t) + \frac{V_0 \, R}{(\omega \, L)^2 + R^2} \sin(\omega t) \quad \cdots\cdots \quad (2)$$

**( 7 marks )**

(ii) Convert $I(t)$ in eq.(2) into the form of $I(t) = I_m \sin(\omega t + \phi)$ and deduce that

$$I_m = \frac{V_0}{\sqrt{(\omega \, L)^2 + R^2}} \quad \& \quad \phi = -\tan^{-1}\left(\frac{\omega \, L}{R}\right) \quad \textit{and thus}$$

$$I(t) = \frac{V_0}{\sqrt{(\omega \, L)^2 + R^2}} \sin\left(\omega t - \tan^{-1}\left(\frac{\omega \, L}{R}\right)\right) \quad \cdots\cdots \quad (3)$$

**( 5 marks )**

## Question five (continued)

(b) (i) Substituting $V_{in}(t)$ & $I(t)$ by $V_0\, e^{i\omega t}$ & $\hat{I}\, e^{i\omega t}$ respectively into eq.(1) to solve for $\hat{I}$ in terms of $V_0$ , $R$ , $L$ & $\omega$ and show that

$$\hat{I} = \frac{V_0}{R + i\,\omega\, L} \quad \cdots\cdots \quad (4)$$

**( 4 marks )**

(ii) Convert $\hat{I}$ in eq.(4) into its polar form, i.e., $\hat{I} \equiv I_C\, e^{i\,\phi_C}$ , deduce that

$$I_C = \frac{V_0}{\sqrt{\omega^2\, L^2 + R^2}} \quad \cdots\cdots \quad (5)$$

$$\phi_C = -\tan^{-1}\left(\frac{\omega\, L}{R}\right) \quad \cdots\cdots \quad (6)$$

**( 3 marks )**

(c) (i) The gain G of any AC circuit is defined as $G \equiv \frac{|V_{out}(t)|}{|V_{in}(t)|} \xrightarrow{\text{here is}} \frac{\left|L\,\frac{d\,I(t)}{d\,t}\right|}{V_0}$ ,

deduce that for our given circuit

$$G = \frac{1}{\sqrt{\left(\frac{R}{\omega\, L}\right)^2 + 1}} \quad \cdots\cdots \quad (7)$$

**( 3 marks )**

(ii) What are the limiting values for G in the cases of high and of low frequency? Then explain briefly why this circuit serves as a high-pass filter. **( 3 marks )**

$$\vec{\nabla}\times\vec{F}=\vec{e}_x\left(\frac{\partial(F_z)}{\partial y}-\frac{\partial(F_y)}{\partial z}\right)+\vec{e}_y\left(\frac{\partial(F_x)}{\partial z}-\frac{\partial(F_z)}{\partial x}\right)+\vec{e}_z\left(\frac{\partial(F_y)}{\partial x}-\frac{\partial(F_x)}{\partial y}\right)$$

$$=\frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial\phi}-\frac{\partial(F_\phi\,\rho)}{\partial z}\right)+\vec{e}_\phi\left(\frac{\partial(F_\rho)}{\partial z}-\frac{\partial(F_z)}{\partial\rho}\right)+\frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\,\rho)}{\partial\rho}-\frac{\partial(F_\rho)}{\partial\phi}\right)$$

$$=\frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\,r\sin(\theta))}{\partial\theta}-\frac{\partial(F_\theta\,r)}{\partial\phi}\right)+\frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial\phi}-\frac{\partial(F_\phi\,r\sin(\theta))}{\partial r}\right)+\frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\,r)}{\partial r}-\frac{\partial(F_r)}{\partial\theta}\right)$$

where $\vec{F}=\vec{e}_x\,F_x+\vec{e}_y\,F_y+\vec{e}_z\,F_z=\vec{e}_\rho\,F_\rho+\vec{e}_\phi\,F_\phi+\vec{e}_z\,F_z=\vec{e}_r\,F_r+\vec{e}_\theta\,F_\theta+\vec{e}_\phi\,F_\phi$ and

$$d\vec{l}=\vec{e}_x\,dx+\vec{e}_y\,dy+\vec{e}_z\,dz=\vec{e}_\rho\,d\rho+\vec{e}_\phi\,\rho\,d\phi+\vec{e}_z\,dz=\vec{e}_r\,dr+\vec{e}_\theta\,r\,d\theta+\vec{e}_\phi\,r\sin(\theta)\,d\phi$$