UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

SUPPLEMENTARY EXAMINATION 2012/2013

TITLE OF PAPER : ELECTRICITY AND MAGNETISM

COURSE NUMBER : P221

TIME ALLOWED : THREE HOURS

INSTRUCTIONS : ANSWER ANY FOUR OUT OF FIVE QUESTIONS.
EACH QUESTION CARRIES 25 MARKS.

MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN.

THIS PAPER HAS TEN PAGES, INCLUDING THIS PAGE.

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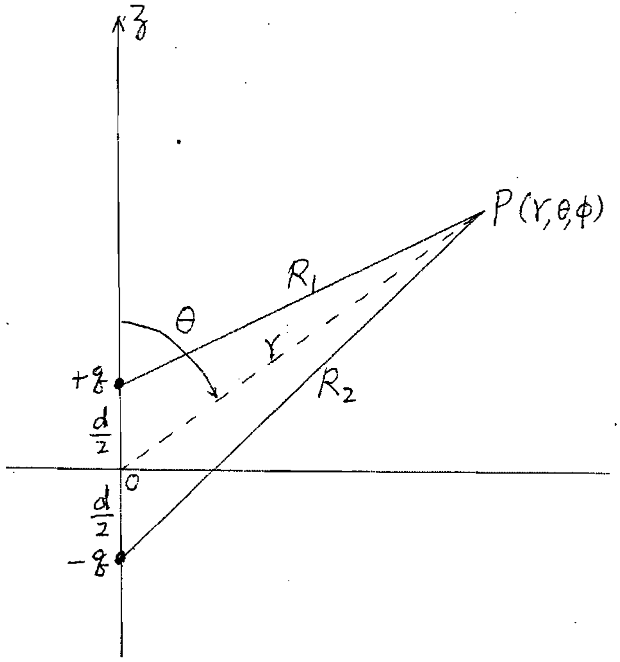
Question one

(a) According to Coulomb's law, the electric field $\vec{E}$ at a space point $P:(r,\theta,\phi)$ produced by a point charge q situated at the origin is $\vec{E}=\vec{e}_r \dfrac{q}{4\pi\varepsilon_0 r^2}$. Using $\Phi=-\int_{P_0}^{P}\vec{E}\bullet d\vec{l}$ where $P_0:(r_0,\theta_0,\phi_0)$ is the zero potential reference point and carrying out the integration to show that its associated electric potential Φ is $\Phi=\dfrac{q}{4\pi\varepsilon_0 r}$ if $r_0\to\infty$

(7 marks)

(b) For an electric dipole situated at the origin , i.e., $+q$ situated at $z=\dfrac{d}{2}$ & $-q$ situated at $z=-\dfrac{d}{2}$ along z – axis , the electric potential Φ at the space point $P:(r,\theta,\phi)$ due to this dipole is $\Phi=\Phi_1+\Phi_2$ where $\Phi_1=\dfrac{+q}{4\pi\varepsilon_0 R_1}$ and $\Phi_2=\dfrac{-q}{4\pi\varepsilon_0 R_2}$, as depicted in the following diagram



(i) for $r>>d$, use $R_1\approx r-\dfrac{d}{2}\cos(\theta)$ and $R_2\approx r+\dfrac{d}{2}\cos(\theta)$, deduce that $\Phi\approx\dfrac{p\cos(\theta)}{4\pi\varepsilon_0 r^2}$ where $p=q\,d$ is the electric dipole moment . **(10 marks)**

(ii) from Φ in (b)(i) , deduce that the electric field at far away point from the electric dipole source situated at the origin given in the above diagram is

$$\vec{E}=\frac{p}{4\pi\varepsilon_0 r^3}\left(\vec{e}_r\,2\cos(\theta)+\vec{e}_\theta\sin(\theta)\right)$$ **(8 marks)**

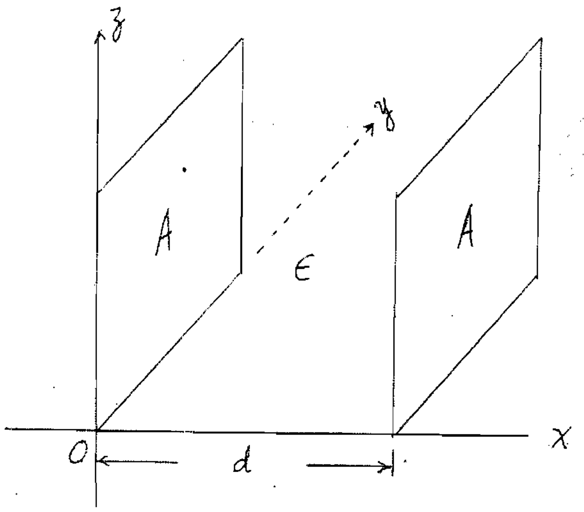
Question two

(a) Consider depositing some free charges of the same sign onto a good conductor.

(i) Explain why these free charges eventually settled on the conductor's surface and the electric field inside the conductor is zero. Also explain why the conductor surface becomes an equal potential surface in electrostatic case? **(2+2 marks)**

(ii) Draw a tiny pill box shape closed surface across the conductor's surface and apply the integral form of electric Gauss law to deduce that the normal component of the electric field at any point on the conductor's surface E_n is related to the surface charge density at the same point ρ_s by the relation of $E_n = \dfrac{\rho_s}{\varepsilon}$ where the normal direction is pointing from the inside toward the outside of the conductor, if the conductor is embedded in a dielectric medium of permittivity ε . **(6 marks)**

(b) Consider a two-parallel conducting thin plate capacitor system as shown in the following diagram



where A is the plate surface area, d is the plate separation and ε is the electric permittivity of the insulating material layer in-between the two plates.

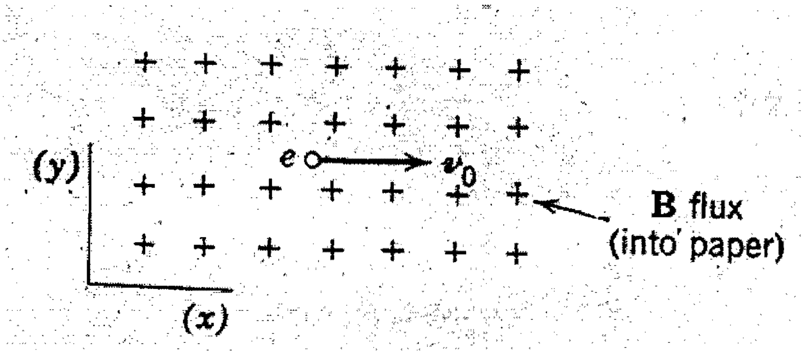
(i) If the total electric charge $-Q$ & $+Q$ are uniformly deposited on the *left* $(x=0)$ & *right* $(x=d)$ conducting plate surfaces facing each other, use the boundary condition in (a) and find the electric field on both conducting surface and show that they are the same. **(6 marks)**

(ii) Assuming this same electric field $\vec{E}$ exist in the regime between two conducting plates, use $V = -\int_{x=0}^{x=d} \vec{E} \bullet d\vec{l}$ to find the potential difference V in between the two plates and thus write down the capacitance C of this system in terms of ε , d & A and show that $C = \dfrac{\varepsilon A}{d}$. **(6 marks)**

(iii) If given the values of $\varepsilon = 4\,\varepsilon_0$, $d = 5\ cm$ & $A = 200\ cm^2$ find the value of C . **(3 marks)**

Question three

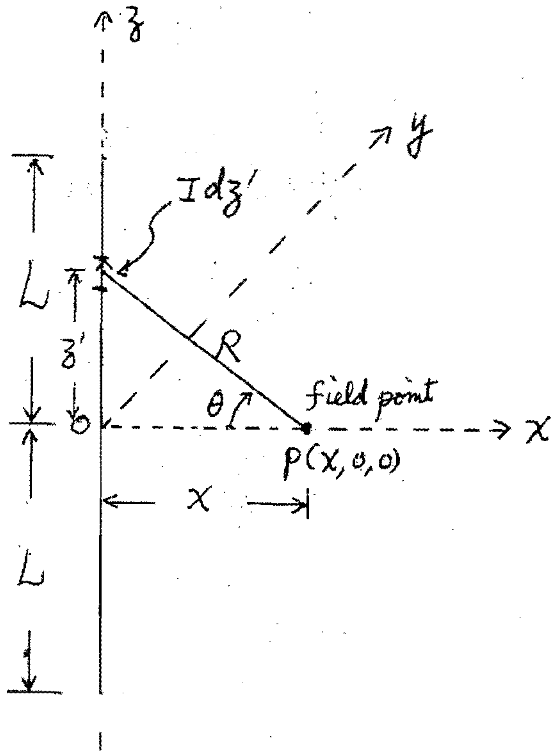
(a) An electron is shot with an initial velocity $\vec{v}_0 = \vec{e}_x\, 10^5 \ \frac{m}{s}$ into a region containing an uniform magnetic field $\vec{B} = -\vec{e}_z\, 10^{-4} \ \frac{Wb}{m^2}$ as shown in the figure below.



(i) Assuming the magnetic field to be uniform over a sufficiently large region of space, show your reasoning as to why this electron should take a circular path, then determine the location and radius of this circle. **(6 marks)**

(ii) Determine what vector electric field intensity $\vec{E}$ will just overcome the effect of the given magnetic field to cause the electron to travel in a straight line along the x axis. **(3 marks)**

(b) A thin conducting wire of length $2\,L$, with its central axis coinciding with the z axis and its centre point coinciding with the origin, carries a steady total current I along positive z direction as shown in the figure below.



Question three (continued)

(i) Since the given current source is only along z axis thus its produced vector potential at a field point P (x , 0 , 0) is also having only z component A_z , i.e.,

$$\vec{A} = \vec{e}_z\ A_z \quad where \quad A_z = \int_{z'=-L}^{z'=+L} \frac{\mu_0\ I\ d z'}{4\ \pi\ R} = 2 \int_{z'=0}^{z'=+L} \frac{\mu_0\ I\ d z'}{4\ \pi\ \sqrt{(z')^2 + x^2}} \ ,$$

carry out the above integral for A_z about z' and show that

$$A_z = \frac{\mu_0\ I}{2\ \pi} \ln\left(\frac{L + \sqrt{L^2 + x^2}}{x}\right)$$ **(10 marks)**

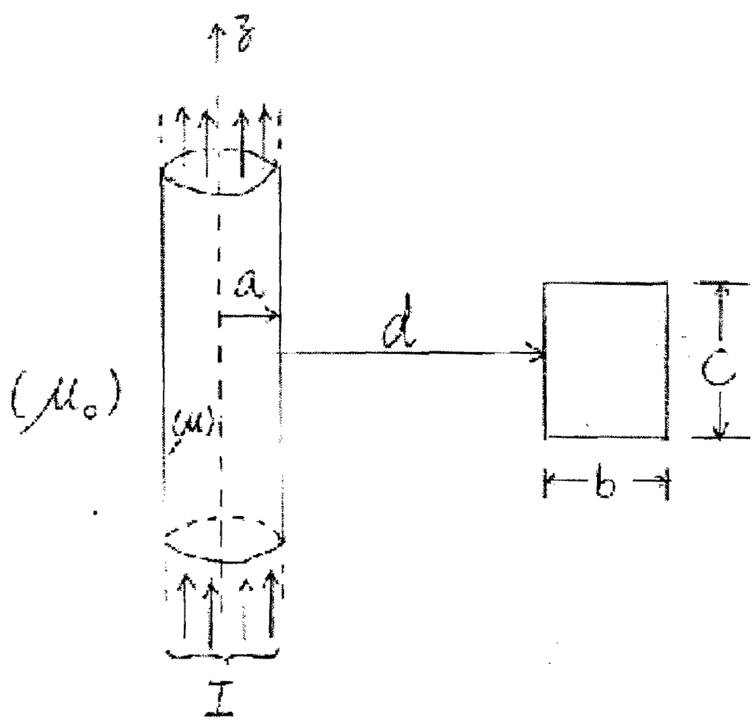
(Hint : set $z' = x \tan(\theta)$, $\int \sec(\theta)\, d\theta = \ln(\sec(\theta) + \tan(\theta))$)

(ii) For $L >> x$, *i.e.*, $\sqrt{L^2 + x^2} \to L$, use $\vec{A} = \vec{e}_z \frac{\mu_0\ I}{2\ \pi} \ln\left(\frac{2\ L}{x}\right)$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ to find the magnetic field $\vec{B}$ at the field point and show that

$$\vec{B} = \vec{e}_y \frac{\mu_0\ I}{2\ \pi\ x} \ .$$ **(6 marks)**

Question four

(a) A very long straight wire of cross section radius a with a permeability μ, with its central axis coinciding with the z – axis, carries an uniform current density in the positive z direction with the total static current of I, i.e., the current density inside the wire is $\dfrac{I}{\pi a^2}$, as shown in the diagram below :



Use the integral Ampere law and draw proper closed loops to find $\vec{B}$ in terms of ρ , a & I for $0 \le \rho \le a$ & $\rho \ge a$ regions . **(12 marks)**

(b) Placing a rectangular conducting loop of dimension $b \times c$ a distance of $d\,(> a)$ away from the central axis of the wire as shown in the diagram in (a) , i.e., the inner region confined by the rectangular loop in clockwise sense is
$S:\ d \le \rho \le d+b$, $0 \le z \le c$ & $d\vec{s} = \vec{a}_\phi\, d\rho\, dz$,

(i) find the total magnetic flux Φ_m passing through the inner region confined by the rectangular loop, i.e., $\Phi_m = \int_S \vec{B} \bullet d\vec{s}$, in terms of μ_0 , b , c , d & I . Also write down the mutual inductance M between the given rectangular loop and the long straight wire and show that $M = \dfrac{\mu_0}{2\pi} \times \ln\left(\dfrac{d+b}{d}\right) \times c$. **(6 marks)**

(ii) If given the values of $b = 5\ cm$, $c = 8\ cm$ & $d = 10\ cm$ find the value of M . Further if the wire carries a sinusoidal current $I(t) = 2\sin(9t)\ A$ instead of carrying a static current I , find the induced e.m.f. in the rectangular conducting loop . **(7 marks)**

Question five

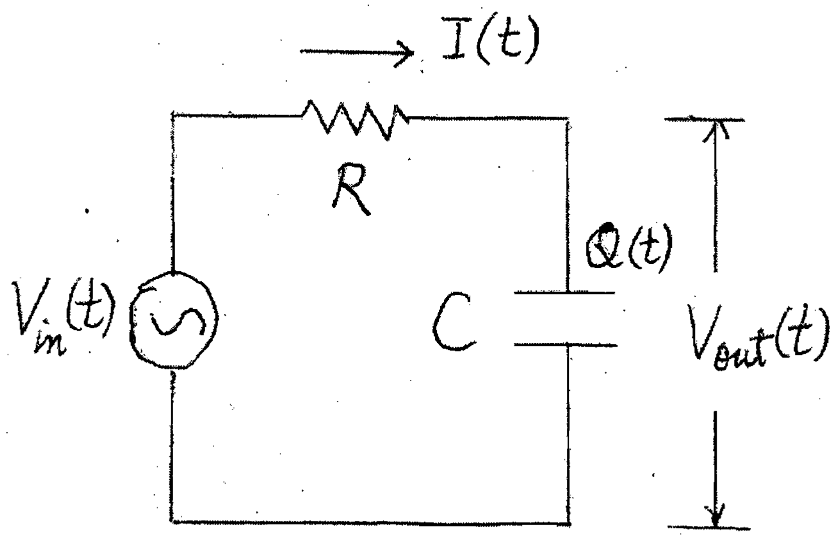
(a) When a AC voltage source $V(t) = V_0 \sin(\omega t)$ is connected to a RLC circuit, be it in parallel or in series, this leads to a sinusoidal current flowing through each circuit element in a form of $I_{element}(t) = I_{element} \sin(\omega t + \phi_{element})$.

(i) For positive $\phi_{element}$ value case, draw $I_{element}(t)$ *vs.* (ωt) diagram for at least one period and indicate/label $I_{element}$ & $\phi_{element}$ on your drawn diagram.

(3 marks)

(ii) Since $I_{element}(t) = I_{element} \sin(\omega t + \phi_{element})$ is the imaginary part of $\hat{I}_{element}(t) = \hat{I}_{element}\, e^{i\omega t}$ *where* $i = \sqrt{-1}$, then what is $\hat{I}_{element}$ in terms of $I_{element}$ & $\phi_{element}$ and also what is the main advantage of using complex scheme to solve sinusoidal driving voltage source circuit problems? **(4 marks)**

(b) Connect an AC voltage source $V_{in}(t) = V_0 \sin(\omega t)$ to a RC circuit in series as shown below:



The circuit equation is $V_{in}(t) = R\,\dfrac{dQ(t)}{dt} + \dfrac{Q(t)}{C}$ ...... (1) *where* $I(t) = \dfrac{dQ(t)}{dt}$.

(i) Substituting $V_{in}(t)$ & $Q(t)$ by $V_0\, e^{i\omega t}$ & $\hat{Q}\, e^{i\omega t}$ respectively into eq.(1) to solve for $\hat{Q}$ in terms of V_0 , R , C & ω and show that

$$\hat{Q} = \frac{V_0}{i\,\omega R + \dfrac{1}{C}} \quad \cdots\cdots \quad (2)$$

(4 marks)

Question five (continued)

(ii) Convert $\hat{Q}$ in eq.(2) into its polar form, i.e., find the magnitude and phase angle of $\hat{Q}$ in terms of V_0 , R , C & ω and show that

$\hat{Q} \equiv Q_C \, e^{i\,\phi_C}$ *where*

$$Q_C = \frac{V_0}{\sqrt{\omega^2 \, R^2 + \dfrac{1}{C^2}}} \quad \cdots\cdots \quad (3)$$

$$\phi_C = -\tan^{-1}(\omega \, R \, C) \quad \cdots\cdots \quad (4)$$

(5 marks)

(iii) The gain of any AC circuit G is defined as $G \equiv \dfrac{|V_{out}(t)|}{|V_{in}(t)|} \xrightarrow{\text{here is}} \dfrac{\left(\dfrac{Q_C}{C}\right)}{V_0}$,

deduce that for our given circuit

$$G = \frac{1}{\sqrt{(2\,\pi\, f\, R\, C)^2 + 1}} \quad \cdots\cdots \quad (5) \qquad \text{where} \quad \omega = 2\,\pi\, f$$

(4 marks)

(iv) Calculate the values of G for the two quite different input frequencies $f = 600 \;\; Hz$ & $600 \;\; kHz$ respectively if given $R = 100 \;\; \Omega$ & $C = 5 \;\; \mu F$. Compare these two results and explain why this circuit serves as a low-pass filter.

(5 marks)

Useful informatios

$e = 1.6 \times 10^{-19}\ C$

$m_e = 9.1 \times 10^{-31}\ kg$

$\mu_0 = 4\,\pi \times 10^{-7}\ \frac{H}{m}$

$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{F}{m}$

$$\oiint_S \vec{E} \bullet d\vec{s} = \frac{1}{\varepsilon} \iiint_V \rho_v\, dv$$

$$\oiint_S \vec{B} \bullet d\vec{s} \equiv 0$$

$$\oint_L \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\iint_S \vec{B} \bullet d\vec{s} \right)$$

$$\oint_L \vec{B} \bullet d\vec{l} = \mu \iint_S \vec{J} \bullet d\vec{s} + \mu\,\varepsilon \frac{\partial}{\partial t}\left(\iint_S \vec{E} \bullet d\vec{s} \right)$$

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu\,\vec{J} + \mu\,\varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{J} = \sigma\,\vec{E}$$

$\oiint_S \vec{F} \bullet d\vec{s} \equiv \oiiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ *divergence theorem*

$\oint_L \vec{F} \bullet d\vec{l} \equiv \iint_S \left(\vec{\nabla} \times \vec{F}\right) \bullet d\vec{s}$ *Stokes' theorem*

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} f\right) \equiv 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{F}\right) \equiv \vec{\nabla}\left(\vec{\nabla} \bullet \vec{F}\right) - \nabla^2 \vec{F}$$

$$\vec{\nabla} f = \vec{e}_x \frac{\partial f}{\partial x} + \vec{e}_y \frac{\partial f}{\partial y} + \vec{e}_z \frac{\partial f}{\partial z} = \vec{e}_\rho \frac{\partial f}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial f}{\partial \phi} + \vec{e}_z \frac{\partial f}{\partial z}$$
$$= \vec{e}_r \frac{\partial f}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{\partial(F_x)}{\partial x} + \frac{\partial(F_y)}{\partial y} + \frac{\partial(F_z)}{\partial z} = \frac{1}{\rho}\frac{\partial(F_\rho\,\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(F_\phi)}{\partial \phi} + \frac{\partial(F_z)}{\partial z}$$
$$= \frac{1}{r^2}\frac{\partial(F_r\,r^2)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\theta \sin(\theta))}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial(F_\phi)}{\partial \phi}$$

$$\vec{\nabla}\times\vec{F}=\vec{e}_x\left(\frac{\partial(F_z)}{\partial y}-\frac{\partial(F_y)}{\partial z}\right)+\vec{e}_y\left(\frac{\partial(F_x)}{\partial z}-\frac{\partial(F_z)}{\partial x}\right)+\vec{e}_z\left(\frac{\partial(F_y)}{\partial x}-\frac{\partial(F_x)}{\partial y}\right)$$

$$=\frac{\vec{e}_\rho}{\rho}\left(\frac{\partial(F_z)}{\partial \phi}-\frac{\partial(F_\phi\,\rho)}{\partial z}\right)+\vec{e}_\phi\left(\frac{\partial(F_\rho)}{\partial z}-\frac{\partial(F_z)}{\partial \rho}\right)+\frac{\vec{e}_z}{\rho}\left(\frac{\partial(F_\phi\,\rho)}{\partial \rho}-\frac{\partial(F_\rho)}{\partial \phi}\right)$$

$$=\frac{\vec{e}_r}{r^2\sin(\theta)}\left(\frac{\partial(F_\phi\,r\sin(\theta))}{\partial\theta}-\frac{\partial(F_\theta\,r)}{\partial\phi}\right)+\frac{\vec{e}_\theta}{r\sin(\theta)}\left(\frac{\partial(F_r)}{\partial\phi}-\frac{\partial(F_\phi\,r\sin(\theta))}{\partial r}\right)+\frac{\vec{e}_\phi}{r}\left(\frac{\partial(F_\theta\,r)}{\partial r}-\frac{\partial(F_r)}{\partial\theta}\right)$$

where $\vec{F}=\vec{e}_x\,F_x+\vec{e}_y\,F_y+\vec{e}_z\,F_z=\vec{e}_\rho\,F_\rho+\vec{e}_\phi\,F_\phi+\vec{e}_z\,F_z=\vec{e}_r\,F_r+\vec{e}_\theta\,F_\theta+\vec{e}_\phi\,F_\phi$ and

$d\vec{l}=\vec{e}_x\,dx+\vec{e}_y\,dy+\vec{e}_z\,dz=\vec{e}_\rho\,d\rho+\vec{e}_\phi\,\rho\,d\phi+\vec{e}_z\,dz=\vec{e}_r\,dr+\vec{e}_\theta\,r\,d\theta+\vec{e}_\phi\,r\sin(\theta)\,d\phi$