UNIVERSITY OF SWAZILAND

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF PHYSICS

MAIN EXAMINATION 2012/2013

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| TITLE OF PAPER | : | MATHEMATICAL METHODS FOR PHYSICISTS |
| COURSE NUMBER | : | P272 |
| TIME ALLOWED | : | THREE HOURS |
| INSTRUCTIONS | : | ANSWER ANY FOUR OUT OF SIX QUESTIONS. EACH QUESTION CARRIES 25 MARKS.<br><br>MARKS FOR DIFFERENT SECTIONS ARE SHOWN IN THE RIGHT-HAND MARGIN. |

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## P272 MATHEMATICAL METHODS FOR PHYSICIST

### Question one

(a) Given a vector field in Cartesian coordinates as $\vec{F} = \vec{e}_x\, 5\, x\, y\, z - \vec{e}_z\, 4\, x^2\, y$ , show that it satisfies the following vector identity $\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{F}\right) \equiv 0$ . **( 6 marks )**

(b) Given $\vec{F} = \vec{e}_x \left(-6\, x\, z\right) + \vec{e}_y \left(6\, y^2\right) + \vec{e}_z \left(-3\, x^2\right)$ find the value of the following line integral

$\int_{P_1,\, L}^{P_2} \vec{F} \bullet d\vec{l}$ if $P_1 : (-2, -2, 1)$ , $P_2 : (2, 2, 1)$ and

(i) L : a straight line from $P_1$ to $P_2$ on $z = 1$ plane . **( 6 marks )**

(ii) L : a semi-circular path in counter clockwise sense from $P_1$ to $P_2$ on $z = 1$ plane, centred at ( 0 , 0 , 1 ) with a radius of $2\sqrt{2}$ .

Compare this answer with that obtained in (b)(i) and show that the given $\vec{F}$ is a conservative vector field. **( 10 marks )**

(Hint : L : $z = 1$ , $x = 2\sqrt{2}\cos(t)$ , $y = 2\sqrt{2}\sin(t)$ & t from $\pi + \dfrac{\pi}{4}$ to $\dfrac{\pi}{4}$)

(iii) Show that the associated scalar potential $f$ of this given conservative vector field is $f = 2\, y^3 - 3\, x^2\, z$ . **( 3 marks )**

**Question two**

Given $\vec{F} = \vec{e}_\rho \left(4\rho^3\right) + \vec{e}_\phi \left(\rho^3 \sin\phi\right) + \vec{e}_z \left(2\rho z^2\right)$ in cylindrical coordinates,

(a) find the value of $\oint_S \vec{F} \bullet d\vec{s}$ if $S$ is the closed surface enclosing the cylindrical tube of cross-sectional radius 5 and tube height 9 , i.e., $S = S_1 + S_2 + S_3$ where

$S_1$ : $\left(z = -3 \ , \ 0 \le \rho \le 5 \ , \ 0 \le \phi \le 2\pi \ \ \& \ \ d\vec{s} = -\vec{e}_z \ \rho \, d\rho \, d\phi \right)$

$S_2$ : $\left(z = +6 \ , \ 0 \le \rho \le 5 \ , \ 0 \le \phi \le 2\pi \ \ \& \ \ d\vec{s} = +\vec{e}_z \ \rho \, d\rho \, d\phi \right)$

$S_3$ : $\left(\rho = 5 \ , \ 0 \le \phi \le 2\pi \ , \ -3 \le z \le +6 \ \ \& \ \ d\vec{s} = \vec{e}_\rho \ \rho \, d\phi \, dz \xrightarrow{\rho = 5} \vec{e}_\rho \ 5 \, d\phi \, dz \right)$

**( 13 marks )**

(b) (i) find $\vec{\nabla} \bullet \vec{F}$ , **( 4 marks )**

(ii) then evaluate the value of $\iiint_V \left(\vec{\nabla} \bullet \vec{F}\right) dv$ where V is bounded by S given in (a) , i.e., V : $0 \le \rho \le 5 \ , \ 0 \le \phi \le 2\pi \ , \ -3 \le z \le +6 \ \ \& \ \ dv = \rho \, d\rho \, d\phi \, dz$ . Compare this value with that obtained in (a) and make a brief comment.

**( 8 marks )**

## Question three

Given the following non-homogeneous differential equation as

$$\frac{d^2\,x(t)}{d\,t^2} + 2\,\frac{d\,x(t)}{d\,t} + 5\,x(t) = f(t)\ ,$$

(a) find its particular solution $x_p(t)$ if

(i) $f(t) = 26\,\sin(3\,t)$ , **( 6 marks )**

(ii) $f(t) = 5\,t$ , **( 5 marks )**

(b) for the homogeneous part of the given non-homogeneous differential equation , i.e., $\frac{d^2\,x(t)}{d\,t^2} + 2\,\frac{d\,x(t)}{d\,t} + 5\,x(t) = 0$ , set $x(t) = e^{\alpha\,t}$ and find the appropriate values of $\alpha$ and thus write down its general solution $x_h(t)$ **( 5 marks )**

(c) if $f(t) = 26\,\sin(3\,t)$, write down the general solution of the given non-homogeneous differential equation in terms of the answers obtained in (a) & (b) . If the initial conditions are $x(0) = 2$ & $\left.\frac{d\,x(t)}{d\,t}\right|_{t=0} = 1$ , find its specific solution $x_s(t)$ . **( 9 marks )**

**Question four**

(a) Given the following 2-D Laplace equation in spherical coordinates as

$$\nabla^2 f(r,\theta)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f(r,\theta)}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f(r,\theta)}{\partial\theta}\right)=0 \quad \text{, set}$$

$f(r,\theta)=F(r)\,G(\theta)$ and use separation variable scheme to separate the above partial differential equation into two ordinary differential equations. **( 5 marks )**

(b) Given a Bessel's differential equation as :

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}+\left(x^2-9\right)y(x)=0 \text{ , set } y(x)=\sum_{n=0}^{\infty}a_n\,x^{n+s} \quad \& \quad a_0\neq 0 \text{ and utilize}$$

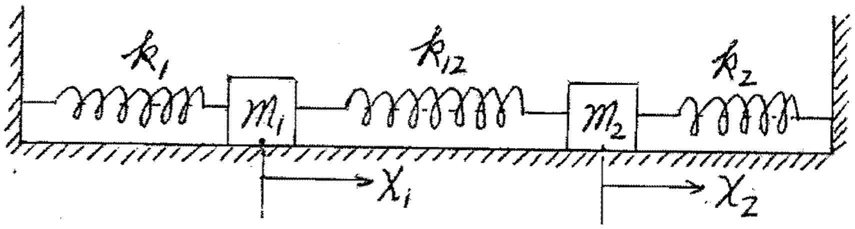
the power series method,

(i) write down its indicial equations and show that $s=-3 \; or \; +3$ and $a_1=0$ , **( 8 marks )**

(ii) write down its recurrence relation. Set $a_0=1$ and use the recurrence relation to generate two independent solutions in power series form truncated up to $a_6$ term. Show that one of the independent solution is a divergent series. **( 12 marks )**

**Question five**

Two simple harmonic oscillators are joined by a spring with a spring constant $k_{12}$ as shown in the diagram below :




The equations of motion for this coupled oscillator system ignoring friction are given as

$$\begin{cases} m_1 \dfrac{d^2 x_1(t)}{dt^2} = -\left(k_1 + k_{12}\right) x_1(t) + k_{12}\, x_2(t) \\ m_2 \dfrac{d^2 x_2(t)}{dt^2} = k_{12}\, x_1(t) - \left(k_2 + k_{12}\right) x_2(t) \end{cases}$$

where $x_1$ & $x_2$ are horizontal displacements of $m_1$ & $m_2$ measured from their respective resting positions.

If given $m_1 = 1\ kg$ , $m_2 = 3\ kg$ , $k_1 = 2\ \dfrac{N}{m}$ , $k_2 = 6\ \dfrac{N}{m}$ & $k_{12} = 9\ \dfrac{N}{m}$ ,

(a) set $x_1(t) = X_1\, e^{i\omega t}$ & $x_2(t) = X_2\, e^{i\omega t}$ , then the above given equations can be deduced to the following matrix equation $A\,X = -\,\omega^2\, X$ where

$$A = \begin{pmatrix} -11 & 9 \\ 3 & -5 \end{pmatrix} \quad \& \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ **( 5 marks )**

(b) find the eigenfrequencies $\omega$ of the given coupled system , **( 6 marks )**

(c) find the eigenvectors X of the given coupled system corresponding to each eigenfrequencies found in (b), **( 6 marks )**

(d) find the normal coordinates of the given coupled system , **( 6 marks )**

(e) write down the general solutions for $x_1(t)$ & $x_2(t)$ . **( 2 marks )**

**Question six**

An elastic string of length 8 is fixed at its two ends, i.e., at $x = 0$ & $x = 8$ and its transverse deflection $u(x,t)$ satisfies the following one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 4\,\frac{\partial^2 u(x,t)}{\partial x^2} \quad ,$$

(a) use separation of variable scheme to split the above partial differential equation into two ordinary differential equations and then write down the general solution of $u(x,t)$.

**( 8 marks )**

(b) given the general solution of $u(x,t)$ , after satisfying two fixed end conditions as well as zero initial speed condition, as $u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{8}\right)\cos\left(\frac{n\pi t}{4}\right)$ where $E_n$ $n = 1,2,3,\cdots\cdots$ are arbitrary constants , then find $E_n$ in terms of $n$ and calculate the values of $E_1$ , $E_2$ & $E_3$ if the initial position of the string , i.e., $u(x,0)$ , is given as $u(x,0) = \begin{cases} 3x & if \quad 0 \le x \le 2 \\ -x+8 & if \quad 2 \le x \le 8 \end{cases}$

( hint : $\int_{x=0}^{8} \sin\left(\frac{n\pi x}{8}\right)\sin\left(\frac{m\pi x}{8}\right)dx = \begin{cases} 0 & if \quad n \ne m \\ 4 & if \quad n = m \end{cases}$ &

$\int x \sin\left(\frac{n\pi x}{8}\right)dx = \frac{64}{n^2\pi^2}\sin\left(\frac{n\pi x}{8}\right) - \frac{8}{n\pi}\,x\cos\left(\frac{n\pi x}{8}\right)$ ) **( 17 marks )**

## Useful informations

The transformations between rectangular and spherical coordinate systems are :

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases} \quad \& \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The transformations between rectangular and cylindrical coordinate systems are :

$$\begin{cases} x = \rho\cos(\phi) \\ y = \rho\sin(\phi) \\ z = z \end{cases} \quad \& \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

$$\vec{\nabla} f = \vec{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial u_1} + \vec{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial u_2} + \vec{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{h_1\, h_2\, h_3}\left(\frac{\partial(F_1\, h_2\, h_3)}{\partial u_1} + \frac{\partial(F_2\, h_1\, h_3)}{\partial u_2} + \frac{\partial(F_3\, h_1\, h_2)}{\partial u_3}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\vec{e}_1}{h_2\, h_3}\left(\frac{\partial(F_3\, h_3)}{\partial u_2} - \frac{\partial(F_2\, h_2)}{\partial u_3}\right) + \frac{\vec{e}_2}{h_1\, h_3}\left(\frac{\partial(F_1\, h_1)}{\partial u_3} - \frac{\partial(F_3\, h_3)}{\partial u_1}\right) + \frac{\vec{e}_3}{h_1\, h_2}\left(\frac{\partial(F_2\, h_2)}{\partial u_1} - \frac{\partial(F_1\, h_1)}{\partial u_2}\right)$$

where $\vec{F} = \vec{e}_1\, F_1 + \vec{e}_2\, F_2 + \vec{e}_3\, F_3$ and

| | | | |
|---|---|---|---|
| $(u_1, u_2, u_3)$ | represents | $(x, y, z)$ | for rectangular coordinate system |
| | represents | $(\rho, \phi, z)$ | for cylindrical coordinate system |
| | represents | $(r, \theta, \phi)$ | for spherical coordinate system |
| $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ | represents | $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ | for rectangular coordinate system |
| | represents | $(\vec{e}_\rho, \vec{e}_\phi, \vec{e}_z)$ | for cylindrical coordinate system |
| | represents | $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ | for spherical coordinate system |
| $(h_1, h_2, h_3)$ | represents | $(1, 1, 1)$ | for rectangular coordinate system |
| | represents | $(1, \rho, 1)$ | for cylindrical coordinate system |
| | represents | $(1, r, r\sin(\theta))$ | for spherical coordinate system |

$$\int (t\sin(k\,t))\,dt = -\frac{t\cos(k\,t)}{k} + \frac{\sin(k\,t)}{k^2}$$

$$\int (t\cos(k\,t))\,dt = \frac{t\sin(k\,t)}{k} + \frac{\cos(k\,t)}{k^2}$$